Preliminary Exercises for Factoring Quadratics

In order to factor quadratic polynomials in one variable quickly one must have a facility with a specific number trick—that of quickly finding factors of a number whose sum or difference is given.

**Example 1:** Find the factors of 42 that total 17. With enough practice a person can just look at this problem and almost instantly reply with 3 and 14. In lieu of such practice, here is a methodical approach.

First list the 2-factor factorizations of 42, starting with 1 times 42.

\[
\begin{align*}
1 \times 42 & \\
2 \times 21 & \\
3 \times 14 & \\
6 \times 7 & 
\end{align*}
\]

To establish this list, first write 1 times the number. Then move to 2 and ask yourself: Does 2 go into the number? If yes, write the factorization. If no, skip to the next number. In the example above 2 and 3 did go into the number and were listed, but 4 and 5 did not, so they were skipped. 6 did go into it 7 times, and there were no numbers between 6 and 7 that would go into 42, so that is where the list stopped. Once the list is written, find the totals.

\[
\begin{align*}
1 \times 42 & \quad 1 + 42 = 43 \\
2 \times 21 & \quad 2 + 21 = 23 \\
3 \times 14 & \quad 3 + 14 = 17 
\end{align*}
\]
We really did not need to write the whole list. The answer was the third pair down. If we had suspected that it would be toward the bottom of the list we could have started there, but that involves the difficulty of finding factors close to the square root of the number. Looking at 42, you estimate the square root as 6+. Backing down from 6+ we find 6 as the first factor of 42, but 6+7 is not 17, so you continue down to 5, then 4, then 3, and 3 and 14 work.

**Example 2:** Find the factors of 120 that differ by 7.

\[
\begin{align*}
1 \times 120 \\
2 \times 60 \\
3 \times 40 \\
4 \times 30 \\
5 \times 24 \\
6 \times 20 \\
8 \times 15 \\
10 \times 12
\end{align*}
\]

Difference means subtraction. So:

\[
\begin{align*}
120 - 1 &= 119 \\
60 - 2 &= 58 \\
40 - 3 &= 37 \\
30 - 4 &= 26 \\
24 - 5 &= 19 \\
20 - 6 &= 14 \\
15 - 8 &= 7 \\
12 - 10 &= 2
\end{align*}
\]

Again, with practice comes intuition, and with a large number like 120 a difference of 7 for its factors is rather small. One with intuition would start with the square root of 120 which is about 11. 11 does not divide 120, so move down to 10. 10 does divide 120, 12 times. That difference is 2, not 7, so continue downward. 9 does not divide 120, so skip it and move down to 8. 8 divides 120, 15 times, and the difference of 15 and 8 is 7. The answer is then 8 and 15.
**Example 3:** Find factors of 144 that total 25.

The square root of 144 is 12, and 12 plus 12 is 24, close to 25, so one would start at that end of the list. The answer is not 12, so move down to 11 and 10 which do not divide 144, and then 9 which does divide 144. 9 divides it 16 times. 9 and 16 total 25, so they are the answer.

**Example 4:** Find factors of 72 that differ by 34.

34 is a large difference for a number like 72, so one would start at the top of the factorization list. The answer is obviously not 1 and 72, so one would move up to 2. 2 goes into 72. It goes 36 times. 36 and 2 differ by 34, so 2 and 36 are the factors sought.

There is a more complicated variation of this number game that is also involved in factoring quadratic polynomials. It involves finding factors of two numbers whose cross products in some order have a certain sum or difference.

**Example 5:** Find ordered factors of 6 and 4 whose cross products differ by 5.

Again, with acquired intuition one quickly sees that the 6 factors into 2 and 3 and the 4 into 4 and 1. Again, without the intuition coming from much practice one must be methodical and list all possibilities in all orders. The list looks like:

\[ 6 = 1 \times 6 \text{ or } 2 \times 3 \text{ and } 4 = 1 \times 4 \text{ or } 2 \times 2 \]

To be methodical, start with the 1 and 6 and try all possible orders of the factors of 4. There are three such possibilities, 1 and 4, 4 and 1, and 2 and 2. Using ordered pair notation to keep track of things, and in the list of four numbers multiplying the outside pair together and the inside pair together, the pursuit looks like:

\[
\begin{align*}
(1,6)(4,1) & \quad 1 \times 1 = 1, \ 6 \times 4 = 24, \ 24 - 1 = 23 \\
(1,6)(1,4) & \quad 1 \times 4 = 4, \ 6 \times 1 = 6, \ 6 - 4 = 2 \\
(1,6)(2,2) & \quad 1 \times 2 = 2, \ 6 \times 2 = 12, \ 12 - 2 = 10
\end{align*}
\]
Next try the 2 and 3 with all possible orderings of the factors of 4.

\[
\begin{align*}
(2,3)(4,1) & \quad 2 \times 1 = 2, \ 3 \times 4 = 12, \ 12 - 2 = 10 \\
(2,3)(1,4) & \quad 2 \times 4 = 8, \ 3 \times 1 = 3, \ 8 - 3 = 5 \\
(2,3)(2,2) & \quad 2 \times 2 = 4, \ 3 \times 2 = 6, \ 6 - 4 = 2
\end{align*}
\]

The answer is the 6 gets factored into 2 and 3 and the 4 into 1 and 4. The 2 factored from the six gets paired with the 4 factored from the 4 and the 3 from the 6 with the 1 from the 4. It would appear that \((2,3)(1,4)\) would be an answer. However, as practice for things to come, the two middle numbers will be switched. Write the answer, switching the middle two numbers from above, as two ordered pairs, the first member in each ordered pair being the factors of either number in either order, and the second member in each ordered pair being the factors of the other number with a special order dependent on the order of the first. Our final answer will be \((2,1)(3,4)\). The first numbers in each ordered pair multiply to give the original 6. The outside numbers give 8, the inside numbers give 3, and their difference is the required 5. The last numbers multiply to give the original 4. (First, Outside, Inside, Last, FOIL) Other possible representations of the answer are \((3,4)(2,1)\), \((1,2)(4,3)\), or \((4,3)(1,2)\). Any of those answers would be correct. What they all have in common is they are sequences of four numbers (ignoring the parentheses) with an inside pair and an outside pair. In each case one of those inside or outside pairs multiplies to give 8 and the other 3, and 8 and 3 differ by 5. Also, the first and third numbers in the sequence are factors of one of the original numbers, and the second and fourth are factors of the other. Writing the numbers in a different order would be wrong, i.e. \((2,4) \ (3,1)\) is wrong. This time the inside pair multiplies to 12 and the outside pair to 2, and the difference is 10 rather than 5.

Something important to notice about the above example is that the only possible differences are 23, 2, 10, and 5. If the problem had asked for factors of 42 that yield a difference of 3 the proper answer would be that the problem is impossible.

**Example 6:** Find ordered factors of 6 and 12 whose cross products total 17.

The factors of 6 could be 1 and 6 or 2 and 3, and the factors of 12 could be 1 and 12 or 2 and 6 or 3 and 4. Again, with practiced intuition one might quickly see that the answer is \((2,3)(3,4)\) or any of its rearrangements that
yield a 9 and an 8. (Part of that intuition would recognize that 17 is odd which
would exclude 2 and 6 from the factors of 12. If both factors of one of the
numbers are even then the cross products would have to be even, and the sum
or difference of two even numbers is even.) For the less intuitive here is the
methodical approach:

Six factors into 1 and 6 or 2 and 3. Twelve factors into 1 and 12, 2 and 6, or 3
and 4. Starting with 1 and 6 there are six possible orders for the factors of 12.

\[
(1,6)(12,1) \quad 1 \times 1 = 1, \quad 6 \times 12 = 72, \quad 72 + 1 = 73
\]

\[
(1,6)(1,12) \quad 1 \times 12 = 12, \quad 6 \times 1 = 6, \quad 12 + 6 = 18
\]

\[
(1,6)(6,2) \quad 1 \times 2 = 2, \quad 6 \times 6 = 36, \quad 36 + 2 = 38
\]

\[
(1,6)(2,6) \quad 1 \times 6 = 6, \quad 6 \times 2 = 12, \quad 12 + 6 = 18
\]

\[
(1,6)(4,3) \quad 1 \times 3 = 3, \quad 6 \times 4 = 24, \quad 24 + 3 = 27
\]

\[
(1,6)(3,4) \quad 1 \times 4 = 4, \quad 6 \times 3 = 18, \quad 18 + 4 = 22
\]

That was all of the possibilities with 1 and 6 for 6, so now try 2 and 3. (A
person comfortable with the odd/even relationships would have looked at
just two of those possibilities. The only way to get an odd sum like 17 is by
adding an odd number to an even number, and odd numbers only come from
multiplying odd numbers. 1 \times 1 and 1 \times 3 were the only possibilities.

Trying 2 and 3 for the six also yields six possible orderings. (2 if we play the
odd numbers game.)

\[
(2,3)(12,1) \quad 2 \times 1 = 2, \quad 3 \times 12 = 36, \quad 36 + 2 = 38
\]

\[
(2,3)(1,12) \quad 2 \times 12 = 24, \quad 3 \times 1 = 3, \quad 24 + 3 = 27
\]

\[
(2,3)(6,2) \quad 2 \times 2 = 4, \quad 3 \times 6 = 18, \quad 18 + 4 = 22
\]

\[
(2,3)(2,6) \quad 2 \times 6 = 12, \quad 3 \times 2 = 6, \quad 12 + 6 = 18
\]

\[
(2,3)(4,3) \quad 2 \times 3 = 6, \quad 3 \times 4 = 12, \quad 12 + 6 = 18
\]

\[
(2,3)(3,4) \quad 2 \times 4 = 8, \quad 3 \times 3 = 9, \quad 8 + 9 = 17
\]

This was an example of very unlucky guessing. All twelve possibilities were
explored before the answer appeared. The answer is some variation of
(2,3)(3,4). If the oddness of 17 had been played only four possible solutions
would have been explored. With much practice a person can look at the 6, 12,
and 17 and quickly see that the 17 is 8 plus 9 and the 9 comes from 3 times 3
and the 8 from 2 times 4.
**Example 7**: Find ordered factors of 12 and 12 whose cross products differ by 17.

There will be 18 possible outcomes for this combination because each number has three different factorizations, 1 and 12, 2 and 6, or 3 and 4, which makes three choices with three choices, i.e. \(3 \times 3 = 9\), and each of those choices has two possibilities because the order can be switched. \(3 \times 3 \times 2 = 18\). Rather than list all 18 possibilities, this time advantage will be taken of the fact that 17 is odd, so an odd number pair must be multiplied. This immediately rules out the 2 and 6 factorizations. Starting with the 1 and 12 for the first number, the 1 must multiply either a 1 or a 3 from the second number. The results are:

- \((1, 12)(1, 1)\) \[1 \times 1 = 1, \ 12 \times 12 = 144, \ 144 - 1 = 143\]
- \((1, 12)(4, 3)\) \[1 \times 3 = 3, \ 12 \times 4 = 48, \ 48 - 3 = 45\]

Letting the first 12 be 3 times 4 the possibilities are:

- \((3, 4)(12, 1)\) \[3 \times 1 = 3, \ 4 \times 12 = 48, \ 48 - 3 = 45\]
- \((3, 4)(4, 3)\) \[3 \times 3 = 9, \ 4 \times 4 = 16, \ 16 - 9 = 7\]

None of the possibilities yielded 17, so the answer is that the required factorization is not possible. Simply stating “Not possible” or “NP” will do.

**Factor spotting tricks**

If a number is even then 2 is a factor.
If a number ends in 5 or 0 then 5 is a factor.
If a number ends in 0 then 10 is a factor.
If the last two digits of a number form a number that is divisible by 4 then 4 is a factor.
If the last three digits of a number form a number that is divisible by 8 then 8 is a factor.
If the sum of the digits of a number is divisible by 3 then 3 is a factor.
If the sum of the digits of a number is divisible by 9 then 9 is a factor.
If a number is even and the sum of its digits is divisible by 3 then 6 is a factor.
If the difference of the sum of the odd numbered digits and the sum of the even numbered digits of a number is divisible by 11 then 11 is a factor.