1.4 The Tangent and Velocity Problem

What is a tangent? A tangent to a curve is a line that touches the curve. It should have the same direction as the curve at the point of contact.

For a circle: A tangent is a line that intersects the circle once and only once.

For more complicated curves: Consider a curve $C$ and a point $P$ on the curve:

Let’s look at a concrete example to see how we might get something like line $t$.

Goal: Find an equation of the tangent line to the curve $y = x^2$ at the point $(1, 1)$.

Recall that in order to find the equation of the line, we have to know the slope of the line (either by using the slope-intercept formula for a line or the point-slope formula for a line). In order to find the slope of a line, we need to know two points that lie on the line. This is a problem for us since we only know one point on the line, the point of tangency $(1, 1)$.

Question: How can we find the slope $m$ of the tangent line to the curve $y = x^2$ at the point $(1, 1)$.

We can compute an approximation to $m$ by choosing a nearby point on the parabola and computing the slope of the secant line.

Example 1. *Find the slope of the secant line of the curve for $x = 1$ and $x = 2$. 
Example 2. Find the slope of the secant line of the curve for \( x = 1 \) and \( x = 1.5 \).

Example 3. Find the slope of the secant line of the curve for \( x = 1 \) and \( x = 1.1 \).

Example 4. Find the slope of the secant line of the curve for \( x = 1 \) and \( x = 1.01 \).

Example 5. Find the slope of the secant line of the curve for \( x = 1 \) and \( x = 1.001 \).

Example 6. Find the slope of the secant line of the curve for \( x = 0 \) and \( x = 1 \).

Example 7. Find the slope of the secant line of the curve for \( x = 0.5 \) and \( x = 1 \).
**Example 8.** Find the slope of the secant line of the curve for \( x = 0.9 \) and \( x = 1 \).

**Example 9.** Find the slope of the secant line of the curve for \( x = 0.99 \) and \( x = 1 \).

**Example 10.** Find the slope of the secant line of the curve for \( x = 0.999 \) and \( x = 1 \).

Summarizing what we have found so far in a table of values (rounding to 3 decimal places), we have

<table>
<thead>
<tr>
<th>( x = 1 ) to ( x = )</th>
<th>0</th>
<th>0.5</th>
<th>0.9</th>
<th>0.99</th>
<th>0.999</th>
<th>1.001</th>
<th>1.01</th>
<th>1.1</th>
<th>1.5</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_{sec} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Question:** Do you notice anything interesting?

So, it appears that \( m = \) _________. If so, then we can use the point-slope form of the equation of a line to write the equation of the tangent line through \((1, 1)\) as
Many functions that occur in science are not described by explicit equations; they are defined by experimental data.

**Example 11.** A cardiac monitor is used to measure the heart rate of a patient after surgery. It compiles the number of heartbeats after \( t \) minutes. When the data in the table are graphed, the slope of the tangent line represents the heart rate in beats per minute.

<table>
<thead>
<tr>
<th>( t ) (min)</th>
<th>36</th>
<th>38</th>
<th>40</th>
<th>42</th>
<th>44</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heartbeats</td>
<td>2530</td>
<td>2661</td>
<td>2806</td>
<td>2948</td>
<td>3080</td>
</tr>
</tbody>
</table>

The monitor estimates this value by calculating the slope of a secant line. Use the data to estimate the patient’s heart rate after 42 minutes using the secant line between the points with the given values of \( t \).

- a) \( t = 36 \) and \( t = 42 \)
- b) \( t = 38 \) and \( t = 42 \)
- c) \( t = 40 \) and \( t = 42 \)
- d) \( t = 42 \) and \( t = 44 \)

What are your conclusions?

In real life, the velocity of a car is not constant. The needle of the speedometer doesn’t stay still for very long. We assume from watching the speedometer that the car has a definite velocity at each moment, but how is this instantaneous velocity defined?

**Example 12.** A ball is dropped from the top of LEX LUTHOR: Drop of Doom, 415 feet above the ground, and its height \( h \) above the ground \( t \) seconds after being dropped is given by \( h(t) = 415 - 16t^2 \). What is the velocity of the ball after 5 seconds?