2.7 Rates of Change in the Natural and Social Sciences

We know that if \( y = f(x) \), then the derivative \( dy/dx \) can be interpreted as the rate of change of \( y \) with respect to \( x \).

*Recall:* If \( x \) changes from \( x_1 \) to \( x_2 \), then the change in \( x \) is

and the corresponding change in \( y \) is

The difference quotient

is the \( \frac{\text{ }}{\text{}} \) over the interval \( [x_1, x_2] \) and can be interpreted as \( \frac{\text{ }}{\text{}} \). Its limit as \( \Delta x \to 0 \) is the derivative \( f'(x) \), which can therefore be interpreted as the \( \frac{\text{ }}{\text{}} \) or the slope of the tangent line at \( P(x_1, f(x_1)) \).

We will look at several applications in:

- Physics
- Chemistry
- Biology
- Economics
- Other Sciences

**Physics**

*Velocity and Acceleration:*

If \( s = f(t) \) is the position function of a particle that is moving in a straight line, then \( \Delta s/\Delta t \) represents the average velocity over a time period \( \Delta t \), and \( v = ds/dt \) represents the instantaneous velocity (the rate of change of displacement with respect to time). The instantaneous rate of change of velocity with respect to time is acceleration: \( a(t) = v'(t) = s''(t) \).
Example 1. The position of a particle is given by the equation
\[ s = f(t) = 4t^3 - 16t^2 + 20t, \quad t \geq 0 \]
where \( t \) is measured in seconds and \( s \) in meters.

a) Find the velocity at time \( t \).

b) What is the velocity after 2 s?

c) When does the particle have a velocity of 20 m/s?

d) At what time is the acceleration 0?

e) Graph the position, velocity and acceleration functions for \( 0 \leq t \leq 2 \).
Linear Density:

The linear density of the rod is the derivative of mass with respect to length.

Example 2. The mass of the part of a metal rod that lies between its left end and a point $x$ meters to the right is $3x^2$ kg. Find the linear density when $x = 1$ m

Currents:

The __________ is the rate at which charge flows through a surface. It is measured in units of charge per unit time (often coulombs per second, called amperes).

Example 3. The quantity of charge $Q$ in coulombs (C) that has passed through a point in a wire up to time $t$ (measured in seconds) is given by $Q(t) = t^3 - 2t^2 + 6t + 2$. Find the current when

a) $t = 0.5$ s

b) At what time is the current lowest?

Other rates of change that are important in physics include power (the rate at which work is done), the rate of heat flow, temperature gradient (the rate of change of temperature with respect to position), and the rate of decay of a radioactive substance in nuclear physics.
A chemical reaction results in the formation of one or more substances (called products) from one or more starting materials (called reactants). Let’s consider the reaction

\[ A + B \rightarrow C \]

where A and B are the reactants and C is the product. The concentration of a reactant A is the number of moles per liter and is denoted by \([A]\). The concentration varies during a reaction, so \([A]\), \([B]\), and \([C]\) are all functions of time \(t\). The rate of reaction is obtained by taking the limit of the average rate of reaction as the time interval \(\Delta t\) approaches 0:

\[ \text{Example 4. If one molecule of the product } C \text{ is formed from one molecule of the reactant } A \text{ and one molecule of the reactant } B, \text{ and the initial concentrations of } A \text{ and } B \text{ have a common value } [A]=[B]=a \text{ moles/L, then} \]

\[ [C]=\frac{a^{2}kt}{akt + 1} \]

where \(k\) is a constant.

\(a)\) Find the rate of reaction at time \(t\).
b) Show that if \( x = [C] \), then
\[
\frac{dx}{dt} = k(a - x)^2
\]

**Compressibility:**

We can consider the rate of change of volume with respect to pressure \( (dV/dP) \). As \( P \) increases, \( V \) decreases, so \( dV/dP < 0 \). The \( \beta \) is defined by introducing a minus sign and dividing this derivative by the volume \( V \):

Thus \( \beta \) measures how fast, per unit volume, the volume of a substance decreases as the pressure on it increases at constant temperature.

**Example 5.** The volume \( V \) (in cubic meters) of a sample of air at \( 25^\circ C \) was found to be related to the pressure \( P \) (in kilopascals) by the equation
\[
V = \frac{5.3}{P}
\]

What is the compressibility when \( P = 50 \) kPa?
Biology

**Population Growth:**

The derivative of the number of individuals in a population with respect to time $t$. We will look at examples like this in Chapter 6.

**Laminar Flow:**

When we consider the flow of blood through a blood vessel, such as a vein or artery, we can model the shape of the blood vessel by a cylindrical tube with radius $R$ and length $l$. Because of friction at the walls of the tube, the velocity $v$ of the blood is greatest along the central axis of the tube and decreases as the distance $r$ from the axis increases until $v$ becomes 0 at the wall.

**Law of Laminar Flow:** Let $R$ be the radius of the blood vessel, $r$ the distance from the axis, $l$ the length of the tube, $\eta$ be the viscosity of the blood and $P$ be the pressure difference between the ends of the tube. Then the velocity of the blood is given by:

$$v = \frac{P}{4 \eta l} R^2 \ln \left( \frac{R}{r} \right)$$

The derivative is the instantaneous rate of change of velocity with respect to $r$:

**Example 6.** Consider a blood vessel with radius 0.01 cm, length 3 cm, pressure difference 3000 dynes/cm$^2$, and viscosity $\eta = 0.027$.

a) Find the velocity of the blood along the centerline $r = 0$, at radius $r = 0.005$, and at the wall $r = R = 0.01$ cm.

b) Find the velocity gradient at $r = 0$, $r = 0.005$, and $r = 0.01$. 


Economics

*Marginal Cost:*

The instantaneous rate of change of cost with respect to the number of items produced, is called the

[Blank] by economists.

Taking $\Delta x = 1$ and $n$ large (so that $\Delta x$ is small compared to $n$), we have

Thus the marginal cost of producing $n$ units is approximately equal to the cost of producing one more
unit [the $(n + 1)$st unit].

**Example 7.** The cost function for production of a commodity is

$$C(x) = 339 + 25x - 0.09x^2 + 0.0004x^3$$

a) Find the marginal cost function.

b) Find and interpret $C'(100)$.

c) Compare $C'(100)$ with the cost of producing the 101st item.

Economists also study marginal demand, marginal revenue, and marginal profit. We will consider these in Chapter 3.
Other Sciences

Rates of change occur in all the sciences.

- A geologist is interested in knowing the rate at which a molten rock cools.
- An engineer wants to know the rate at which water flows into or out of a reservoir.
- An urban geographer is interested in the rate of change of the population density in a city as the distance from the city center increases.
- A meteorologist is concerned with the rate of change of atmospheric pressure with respect to height.
- In learning theory, of particular interest is the rate at which performance improves as time passes.
- In sociology, differential calculus is used in analyzing the spread of rumors (or trends).

All of the applications we have just looked at are special cases of the derivative! A single abstract mathematical concept such as the derivative can have different interpretations in each of the sciences. Thus, we can develop the properties for the mathematical concept and then apply the results to all of the sciences.