3.2 The Mean Value Theorem

**Rolle’s Theorem:** Let $f$ be a function that satisfies the following three hypotheses:

1. $f$ is continuous on the closed interval $[a, b]$.
2. $f$ is differentiable on the open interval $(a, b)$.
3. $f(a) = f(b)$

Then there is a number $c$ in $(a, b)$ such that $f'(c) = 0$.

**Proof:**

**Example 1.** Verify that $f(x) = x^3 - x^2 - 6x + 2$ satisfies the three hypotheses of Rolle’s Theorem on $[0, 3]$. Then find all numbers $c$ that satisfy the conclusion of Rolle’s Theorem.
Example 2. Let $f(x) = \tan x$.  

a) Show that $f(0) = f(\pi)$ but there is not number $c$ in $(0, \pi)$ such that $f'(c) = 0$. 

b) Why does this not contradict Rolle’s Theorem?

Our main use of Rolle’s Theorem is in proving the following important theorem.

**The Mean Value Theorem (MVT):** Let $f$ be a function that satisfies the following hypotheses:

1. $f$ is continuous on the closed interval $[a, b]$. 
2. $f$ is differentiable on the open interval $(a, b)$. 

Then there is a number $c$ in $(a, b)$ such that 

or, equivalently, 

Let’s first look at a figure to justify this theorem.
Proof of MVT:
Example 3. Let $f(x) = x^3 - 3x + 2$.

a) Verify that $f$ satisfies the hypotheses of the MVT on the interval $[-2, 2]$.

b) Find all numbers $c$ that satisfy the conclusion of the MVT.
Example 4. Let \( f(x) = 2 - |2x - 1| \).

a) Show that there is no value of \( c \) such that \( f(2) - f(0) = f'(c)(2 - 0) \).

b) Why does this not contradict the MVT?

The MVT can be interpreted as saying that there is a number at which the instantaneous rate of change is equal to the average rate of change over an interval.

Example 5. At 2:00PM a car’s speedometer reads 30 mi/hr. At 2:10PM it reads 50 mi/hr. Show that at some time between 2:00 and 2:10, the acceleration is exactly 120 mi/hr\(^2\).
The main significance of the MVT is that it enables us to obtain information about a function from information about its derivative.

**Example 6.** Suppose that $f$ satisfies the conditions of the MVT and that $3 \leq f'(x) \leq 5$ for all values of $x$. Show that $18 \leq f(8) - f(2) \leq 30$.

**Example 7.** Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of $x$. How large can $f(2)$ possibly be?
The MVT can be used to establish some basic facts of differential calculus:

If \( f'(x) = 0 \) for all \( x \) in an interval \((a,b)\), then \( f \) is constant on \((a,b)\).

**Proof:**

If \( f'(x) = g'(x) \) for all \( x \) in an interval \((a,b)\), then \( f - g \) is constant on \((a,b)\); that is, \( f(x) = g(x) + c \) where \( c \) is a constant.

**Proof:**