3.4 Limits at Infinity; Horizontal Asymptotes

Recall: If \( x \) approaches \( a \) and the result is that the values of \( y \) become arbitrarily large (positive or negative), then \( x = a \) is a _______________________

In this section, we will investigate _______________________. That is, we will investigate what happens to the function \( y = f(x) \) as \( x \) becomes arbitrarily large.

Example 1. Let \( f(x) = \frac{x^2 - 1}{x^2 + 1} \). Use the table to investigate what happens to \( f(x) \) as \( x \) becomes large.

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-1)</th>
<th>(-10)</th>
<th>(-100)</th>
<th>(-1000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td></td>
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</tbody>
</table>

Example 2. Let \( f(x) = \frac{x^2 - 1}{x^2 + 1} \). Use the graph to investigate what happens to \( f(x) \) as \( x \) becomes large.

Let \( f \) be a function defined on some interval \((a, \infty)\). Then

means that the values of \( f(x) \) can be made arbitrarily close to \( L \) by taking \( x \) sufficiently large.

Let \( f \) be a function defined on some interval \((-\infty, a)\). Then

means that the values of \( f(x) \) can be made arbitrarily close to \( L \) by taking \( x \) sufficiently large negative.
Another notation for \( \lim_{x \to \infty} f(x) = L \) is

Again, the symbol \( \infty \) does not represent a number. The expression \( \lim_{x \to \infty} f(x) = L \) is often read as

Some geometric illustrations of \( \lim_{x \to \infty} f(x) = L \):

\[ \text{Myth: A function cannot cross or touch an asymptote.} \]

\[ \text{Truth: As long as the function passes the VLT, the function can cross or touch any asymptote.} \]

The line \( y = L \) is called a ____________________________ of the curve \( y = f(x) \) if either
Example 3. Find the infinite limits, limits at infinity, and asymptotes for the function $f$ whose graph is shown below.

Example 4. Find $\lim_{x \to \infty} \frac{1}{x}$ and $\lim_{x \to -\infty} \frac{1}{x}$. 
Most of the Limit Laws also hold for limits at infinity.

If \( r > 0 \) is a rational number, then

If \( r > 0 \) is a rational number such that \( x^r \) is defined for all \( x \), then

**Example 5. Evaluate**

\[
\lim_{x \to \infty} \frac{4x^3 + 6x^2 - 2}{2x^3 - 4x + 5}
\]

*Indicate which properties of limits are used at each stage.*
Example 6. Find the horizontal and vertical asymptotes of the function

\[ f(x) = \frac{\sqrt{4x^2 + 3x + 2}}{x - 9} \]
Example 7. Compute \( \lim_{x \to \infty} (\sqrt{x^2 - 2x} - x) \).

Example 8. Evaluate \( \lim_{x \to \infty} \sin \frac{1}{x} \).

Example 9. Find the limit or show that it does not exist.
\[
\lim_{x \to \infty} \frac{1 - x^2}{x^3 - x + 1}
\]
The notation

is used to indicate that the values of $f(x)$ become large as $x$ becomes large. Similar meanings are attached to the following symbols:

Example 10. Find $\lim_{x \to \infty} x^5$ and $\lim_{x \to -\infty} x^5$.

Example 11. Find $\lim_{x \to \infty} x^2 - x^4$.

Example 12. Find $\lim_{x \to \infty} \frac{1 + x^6}{x^4 + 1}$. 
SUMMARY: Let \( f(x) = \frac{p(x)}{q(x)} \), where \( p \) and \( q \) are polynomials.

1. If \( \text{deg}(p) = \text{deg}(q) \),

2. If \( \text{deg}(p) < \text{deg}(q) \),

3. If \( \text{deg}(p) > \text{deg}(q) \),

Precise Definitions:

Let \( f \) be a function defined on some interval \((a, \infty)\). Then

means that for every \( \varepsilon > 0 \) there is a corresponding number \( N \) such that

This says that the values of \( f(x) \) can be made arbitrarily close to \( L \) (within a distance \( \varepsilon \), where \( \varepsilon \) is any positive number) by taking \( x \) sufficiently large (larger than \( N \), where \( N \) depends on \( \varepsilon \)).

Graphically,
Let $f$ be a function defined on some interval $(a, \infty)$. Then

means that for every $\varepsilon > 0$ there is a corresponding number $N$ such that

Graphically,

Similar definitions apply for limits as $x \to -\infty$.

**Example 13.** Find a formula for a function that has vertical asymptotes $x = 1$ and $x = 3$ and horizontal asymptote $y = 1$. 
Example 14. A tank contains 5000 L of pure water. Brine that contains 30 g of salt per liter of water is pumped into the tank at a rate of 25 L/min.

a) Show that the concentration of salt after \( t \) minutes (in grams per liter) is

\[
C(t) = \frac{30t}{200 + t}
\]

b) What happens to the concentration as \( t \to \infty \)?