4.2 The Definite Integral

Recall: A limit of the form

arises when we compute an area and when we try to find the distance traveled by an object. This same type of limit occurs in a wide variety of situations, even when \( f \) is not necessarily a positive function. Thus, we will give limits of this form a name and notation.

If \( f \) is a function defined for \( a \leq x \leq b \), we divide the interval \([a,b]\) into \( n \) subintervals of equal width \( \Delta x = \frac{b-a}{n} \). We let \( x_0 = a, x_1, x_2, ..., x_n = b \) be the endpoints of these subintervals and we let \( x_1^*, x_2^*, ..., x_n^* \) be any ________ in these subintervals, so \( x_i^* \) lies in the \( i \)th subinterval \([x_{i-1}, x_i]\). Then the ________ is provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that \( f \) is ________ on \([a,b]\).

The precise meaning of the limit that defines the integral is as follows:

For every number \( \varepsilon > 0 \) there is an integer \( N \) such that

for every integer \( n > N \) and for every choice of \( x_i^* \) in \([x_{i-1}, x_i]\).

The symbol \( \int \) is called an ________. It is an elongated \( S \). In the notation \( \int_a^b f(x) \, dx \), \( f(x) \) is called the ________ and \( a \) and \( b \) are called the ________; \( a \) is the ________ and \( b \) is the _________. For now, the symbol \( dx \) has no meaning by itself; \( \int_a^b f(x) \, dx \) is all one symbol. The \( dx \) indicates that the independent variable is \( x \). The procedure of calculating an integral is called ________

The definite integral \( \int_a^b f(x) \, dx \) is a NUMBER. Thus, it does not depend on \( x \). We could use any letter in place of \( x \) without changing the value of the integral.

The sum

is called a ________. So the definite integral of an integrable function can be approximated to within any desired degree of accuracy by a Riemann sum.

We know that if \( f \) happens to be positive, then the Riemann sum can be interpreted as a sum of areas of approximating rectangles. Thus, the definite integral \( \int_a^b f(x) \, dx \) can be interpreted as the area under the curve \( y = f(x) \) from \( a \) to \( b \) if \( f(x) \geq 0 \) on \([a,b]\).
If $f$ takes on both positive and negative values, then the Riemann sum is the sum of the areas of the rectangles that lie above the $x$-axis and the negatives of the areas of the rectangles that lie below the $x$-axis. A definite integral can be interpreted as a ________________:

where $A_1$ is the area of the region above the $x$-axis and below the graph of $f$, and $A_2$ is the area of the region below the $x$-axis and above the graph of $f$.

There are situations in which it is advantageous to work with subintervals of unequal width. If the subinterval widths are $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$, we have to ensure that all these widths approach 0 in the limiting process. This happens if the largest width, $\max \Delta x_i$, approaches 0. So in this case, the definite integral becomes

Not all functions are integrable. However, the most commonly occurring functions are integrable.

If $f$ is continuous on $[a, b]$, or if $f$ has only a finite number of jump discontinuities, then $f$ is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x) \, dx$ exists.

If $f$ is integrable, then the value of the integral does not depend on the choices of sample points. To simplify the calculation of the integral, we often take the sample points to be right endpoints. Then $x_i^* = x_i$ and the definition of the integral simplifies as follows.

If $f$ is integrable on $[a, b]$ then

where $\Delta x = \frac{b - a}{n}$ and $x_i = a + i\Delta x$. 


Later, it will be important to recognize limits of sums as integrals.

When we use a limit to evaluate a definite integral, we need to know how to work with sums. The following formulas will be useful:

\[
\begin{align*}
(1) \quad \sum_{i=1}^{n} i &= \\
(2) \quad \sum_{i=1}^{n} i^2 &= \\
(3) \quad \sum_{i=1}^{n} i^3 &= \\
(4) \quad \sum_{i=1}^{n} c &= \\
(5) \quad \sum_{i=1}^{n} ca_i &= \\
(6) \quad \sum_{i=1}^{n} (a_i \pm b_i) &=
\end{align*}
\]

**Example 1.** Use the form of the definition to evaluate \[\int_{1}^{4} (x^2 - 4x + 2) \, dx.\]
Example 2. Express

$$\lim_{n \to \infty} \sum_{i=1}^{n} \left( \frac{\cos x_i}{x_i} \right) \Delta x$$

as an integral on the interval $[\pi, 2\pi]$.

Example 3. a) Evaluate the Riemann sum for $f(x) = \sqrt{4 - x^2}$, taking the sample points to be right endpoints and $a = -2$, $b = 2$, and $n = 4$.

b) Evaluate $\int_{-2}^{2} \sqrt{4 - x^2} \, dx$. 
Example 4. The graph of $g$ is shown.

Estimate $\int_{-2}^{4} g(x) \, dx$ with six subintervals using left endpoints.

Example 5. The table gives the values of a function obtained from an experiment.

<table>
<thead>
<tr>
<th>$x$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>-3.4</td>
<td>-2.1</td>
<td>-0.6</td>
<td>0.3</td>
<td>0.9</td>
<td>1.4</td>
<td>1.8</td>
</tr>
</tbody>
</table>

a) Use the table to estimate $\int_{3}^{9} f(x) \, dx$ using three equal subintervals with right endpoints.

b) If the function is known to be an increasing function, can you say whether your estimates are less than or greater than the exact value of the integral?
Note: We will learn an easier method for the evaluation of integrals in the coming sections so that we can evaluate something like the previous example.

Example 6. Evaluate the following integrals by interpreting each in terms of area.

a) $\int_{0}^{2} \sqrt{4 - x^2} \, dx$

b) $\int_{-2}^{4} \left(\frac{1}{2}x + 4\right) \, dx$
Example 7. The graph of $g$ is shown.

Evaluate each integral by interpreting it in terms of areas.

a) $\int_{0}^{2} g(x) \, dx$

b) $\int_{2}^{6} g(x) \, dx$

c) $\int_{7}^{7} g(x) \, dx$
We often choose the $i$th sample point to be the right endpoint of the $i$th subinterval out of convenience. However, when approximating integrals, it is usually better to let the $i$th sample point be the midpoint of the $i$th subinterval, which we denote by $\bar{x}_i$. Using the Riemann sum with midpoints gives the following approximation.

**Midpoint Rule:**

where

and

**Example 8.** Use the Midpoint Rule with $n = 4$ to approximate $\int_0^{\pi/2} \cos^4 x \, dx$ to four decimal places.
Properties of the Definite Integral:

1. \[ \int_{b}^{a} f(x) \, dx = \]

2. \[ \int_{a}^{a} f(x) \, dx = \]

3. \[ \int_{a}^{b} c \, dx = \]

4. \[ \int_{a}^{b} [f(x) \pm g(x)] \, dx = \]

5. \[ \int_{a}^{b} cf(x) \, dx = \]

6. \[ \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx = \]

Example 9. If \[ \int_{1}^{5} f(x) \, dx = 12 \] and \[ \int_{4}^{5} f(x) \, dx = 3.6, \] find \[ \int_{1}^{4} f(x) \, dx. \]
Example 10. Find $\int_{0}^{5} f(x) \, dx$ if

$$f(x) = \begin{cases} 2 & x < 3 \\ x & x \geq 3 \end{cases}$$

\[\text{Comparison Properties of the Integral: If } a \leq b \text{, then}\]

1. If $f(x) \geq 0$ for $a \leq x \leq b$, then ________________.

2. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then ________________.

3. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

Proofs:
Example 11. Use the third comparison property to estimate \( \int_{0}^{2} (x^3 - 3x + 3) \, dx \).