4.3 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus establishes a connection between the two branches of calculus: differential calculus and integral calculus.

**Differential Calculus**  
Arose from the tangent problem

**Integral Calculus**  
Arose from the area problem

Although these two problems seem to be unrelated, they are actually closely related. They are inverse processes. The Fundamental Theorem of Calculus gives the precise inverse relationship between the derivative and the integral. This relationship was used to develop calculus into a systematic mathematical method.

**Example 1.** Let \( g(x) = \int_0^x f(t) \, dt \), where \( f \) is the function whose graph is shown.

a) Evaluate \( g(0) \) and \( g(6) \).

b) Estimate \( g(x) \) for \( x = 1, 2, 3, 4, \) and 5.
c) Where does \( g \) have a maximum value? Where does it have a minimum value?

d) On what interval is \( g \) increasing? decreasing?

e) Sketch a rough graph of \( g \).

f) Use the graph in part (e) to sketch the graph of \( g'(x) \). Compare with the graph of \( f \).
The Fundamental Theorem of Calculus enables us to compute areas and integrals very easily without having to compute them as limits of sums.

**The Fundamental Theorem of Calculus, Part I (FTCI)** If $f$ is continuous on $[a, b]$, then the function $g$ defined by

\[
g(x) = \int_{a}^{x} f(t) \, dt
\]

is continuous on $[a, b]$ and differentiable on $(a, b)$, and ________________.

Using Leibniz notation for derivatives, we can write FTCI as

**Proof:**
Example 2. Use FTCI to find the derivative of

a) \( g(x) = \int_{1}^{x} (2 + t^4)^5 \, dt \)

b) \( G(x) = \int_{x}^{1} \cos \sqrt{t} \, dt \)

c) \( y = \int_{0}^{x^4} \cos^2 \theta \, d\theta \)

d) \( g(x) = \int_{1+2x}^{1-2x} t \sin t \, dt \)
Example 3. If \( f(x) = \int_0^x (1 - t^2) \cos^2 t \, dt \), on what interval is \( f \) increasing?

Example 4. If \( f(x) = \int_0^{\sin x} \sqrt{1 + t^2} \, dt \) and \( g(y) = \int_3^y f(x) \, dx \), find \( g''(\pi/6) \).
The Fundamental Theorem of Calculus, Part II (FTCII) If $f$ is continuous on $[a,b]$, then

where $F$ is any antiderivative of $f$, that is, a function such that $F' = f$.

FTCII can be written as

**Proof:**

FTCII says that if we know an antiderivative $F$ of $f$, then we can evaluate $\int_a^b f(x) \, dx$ simply by subtracting the values of $F$ at the endpoints of the interval $[a,b]$.

Although the theorem may be surprising at first glance, it becomes plausible if we interpret it in physical terms.

If $v(t)$ is the velocity of an object and $s(t)$ is its position at time $t$, then $v(t) = s'(t)$, so $s$ is an antiderivative of $v$. Recall that we considered an object that always moves in the positive direction and made the guess that the area under the velocity curve is equal to the distance traveled. In symbols:

That is exactly what FTCII says in this context.
Example 5. Evaluate the integral.

a) $\int_{-1}^{1} x^{100} \, dx$

b) $\int_{-\pi}^{\pi} \pi \, dx$

c) $\int_{0}^{2} (y - 1)(2y + 1) \, dy$

d) $\int_{0}^{\pi/4} \sec \theta \tan \theta \, d\theta$
Example 6. Let
\[ f(x) = \begin{cases} 
2 & -2 \leq x \leq 0 \\
4 - x^2 & 0 < x \leq 2 
\end{cases} \]
Evaluate \( \int_{-2}^{2} f(x) \, dx \).

Example 7. What is wrong with the following equation?
\[ \int_{-1}^{2} \frac{4}{x^3} \, dx = -\frac{2}{x^2} \bigg|_{-1}^{2} = \frac{3}{2} \]
Taken together, the two parts of the FTC say that differentiation and integration are inverse processes. Each undoes what the other does.

**Example 8.** Consider \( g(x) = \int_0^x (2 + \sin t) \, dt \).

a) Sketch the area represented by \( g(x) \).

b) Find \( g'(x) \) by using FTCI.

c) Find \( g'(x) \) by evaluating the integral using FTCII and then differentiating.