5.1 Area Between Curves

In Chapter 4, we defined and calculated areas of regions that lie under the graphs of functions. Here, we use integrals to find areas of regions that lie between the graphs of two functions.

Consider the region $S$ that lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$, where $f$ and $g$ are continuous functions and $f(x) \geq g(x)$ for all $x$ in $[a, b]$.

(1) Divide $S$ into $n$ strips of equal width.
(2) Approximate the $i$th strip by a rectangle with base $\Delta x$ and height $f(x^*_i) - g(x^*_i)$.
(3) The Riemann sum

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\text{is an approximation to what we intuitively thing of as the area of } S.
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(4) This approximation becomes better and better as $n \to \infty$.

Thus we define the \underline{_________________________} $A$ of the region $S$ as the limiting value of the sum of the areas of these approximating rectangles.

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We recognize the limit in the above definition as the definite integral \underline{_________________________}. Therefore we have the following formula for area.

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\text{The area } A \text{ of the region bounded by the curves } y = f(x), y = g(x), \text{ and the lines } x = a, x = b, \text{ where } f \text{ and } g \text{ are continuous and } f(x) \geq g(x) \text{ for all } x \text{ in } [a, b], \text{ is}
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Note: \text{ If } g(x) = 0, \text{ } S \text{ is the region under the graph of } f \text{ and our general definition of area reduces to our previous definition.}
In the case where both $f$ and $g$ are positive, it is easy to see why the definition is true:

![Graph showing the region bounded by $y = x^2 + 1$, $y = \sqrt{x}$, and $x = 0$ and $x = 1$.]

**Example 1.** *Find the area of the region bounded above by $y = x^2 + 1$, bounded below by $y = \sqrt{x}$, and bounded on the sides by $x = 0$ and $x = 1$.***

In general, when we set up an integral for an area, it is helpful to sketch the region to identify the top curve $y_T$, the bottom curve $y_B$, and a typical approximating rectangle. Then the area of a typical rectangle is $(y_T - y_B)\Delta x$ and the equation

summarizes the procedure of adding (in a limiting sense) the areas of all the typical rectangles.
Example 2. Find the area of the region enclosed by $y = x^2 - 2x$ and $y = x + 4$.

Example 3. Find the area of the region bounded by the curves $y = |x|$ and $y = x^2 - 2$. 
If we are asked to find the area between the curves \( y = f(x) \) and \( y = g(x) \) where \( f(x) \geq g(x) \) for some values of \( x \) but \( g(x) \geq f(x) \) for other values of \( x \), then we split the given region \( S \) into several regions \( S_1, S_2, \ldots \) with areas \( A_1, A_2, \ldots \). We then define the area of the region \( S \) to be the sum of the areas of the smaller regions \( S_1, S_2, \ldots \), that is \( A = A_1 + A_2 + \ldots \).

**Procedure for Finding the Area Between Two Curves:**

a) Find the intersection points of \( f(x) \) and \( g(x) \).
   - \( f(x) \) and \( g(x) \) intersect when \( f(x) = g(x) \).

b) Sketch the graphs of \( f(x), g(x), x = a \) and \( x = b \) on the same set of axes.

c) Determine the number of subregions (and thus the number of integrals) that you need to find the area for.

d) Set up the integrals of the form:

e) Find the area by evaluating the integrals.

**Example 4.** Find the area of the region bounded by the curves \( y = x^3 \) and \( y = x \).

Some regions are best treated by regarding \( x \) as a function of \( y \). If a region is bounded by curves with equations \( x = f(y) \), \( x = g(y) \), \( y = c \), and \( y = d \), where \( f \) and \( g \) are continuous and \( f(y) \geq g(y) \) for \( c \leq y \leq d \), then its area is

If we write \( x_R \) for the right boundary and \( x_L \) for the left boundary, then we have

Here, a typical approximating rectangles has dimensions \( x_R - x_L \) and \( \Delta y \).
Example 5. Find the area enclosed by $4x + y^2 = 12$ and $x = y$.

Example 6. Find the area enclosed by $x = y^4$ and $x = 2 - y^2$. 
Example 7. The figure shows graphs of the marginal revenue function $R'$ and the marginal cost function $C'$ for a manufacturer. Assume that $R$ and $C$ are measured in thousands of dollars. What is the meaning of the area of the shaded region? Use the Midpoint Rule using four rectangles to estimate the value of this quantity.