5.2 Volumes

Recall that one of the applications of integration is to find the area of a region. Another important application is to find the volume of a three-dimensional solid. Being able to find the volume of a solid is important for many reasons, one being they are commonly used in engineering and manufacturing.

We have an intuitive idea of what volume means, but we must make this idea precise by using calculus to give an exact definition of volume.

Consider a cylinder. (First of all, it is important to note that a cylinder is ANY object that consists of two identical flat ends (called bases) connected by line segments (of equal length) perpendicular to the bases.

In the case of cylinders, finding the volume is “easy”:

**The Disk Method:**

If a region in a plane is revolved about a line, the resulting solid is a ______________________________, and the line is called the ______________________________.

**An Easy Example:** Imagine you revolve a rectangle about an axis adjacent to one of its sides.

a) What solid would be formed?

b) How can we find the volume of this solid?

For a general region:
Example 1. Find the volume of the solid obtained by rotating the region bounded by the curves $y = \sqrt{25 - x^2}$ and $y = 0$ from $x = 2$ to $x = 4$.

An Easy Example, Continued: How can we find the volume of a tube?

This gives a hint for what to do if the resulting solid of revolution has a “hole” in it?

Example 2. Find the volume of the solid of revolution obtained by revolving the region bounded by $y = x$ and $y = x^2$ about the $x$-axis.
We can also revolve regions about the $y$-axis.

**Example 3.** Find the volume of the solid obtained by rotating the region bounded by $y = \frac{1}{4}x^2$, $y = 1$, and $x = 0$ about the $y$-axis.

**Example 4.** Find the volume of the solid of revolution obtained by revolving the region bounded by $y = x$ and $y = x^2$ about the $y$-axis.
It is further possible to rotate about a line other than one of the axes.

**Example 5.** Find the volume of the solid obtained by rotating the region bounded by the curves \( y = \sin x \), \( y = \cos x \), \( x = 0 \), and \( x = \frac{\pi}{4} \) about the line \( y = -1 \).

**Example 6.** Find the volume of the solid obtained by rotating the region bounded by the curves \( y = x^2 \) and \( x = y^2 \) about the line \( x = 1 \).
Volumes of Solids with Known Cross Sections

For a solid $S$ that isn’t a cylinder, we first cut $S$ into pieces (kind of like cutting a loaf of bread into slices) and approximate each piece by a cylinder. We estimate the volume of $S$ by adding the volumes of the cylinders. We arrive at the exact volume of $S$ though a limiting process in which the number of pieces become large.

1. Divide $S$ into $n$ “slabs” of equal width $\Delta x$ by using the planes $P_{x_1}, P_{x_2}, \ldots$ to slice the solid.

2. If we choose sample points $x^*_i$ in $[x_{i-1}, x_i]$, we can approximate the volume of the $i$th slab, $S_i$, by a cylinder with base area $A(x^*_i)$ and “height” $\Delta x$.

3. The volume of this cylinder is $\int x^*_i \Delta x$. So an approximation to the area of $S_i$ is

4. Adding the volumes of these slabs, we get an approximation to the total volume:

This approximation gets better and better as $n \to \infty$. Thus, we define the volume as the limit of these sums as $n \to \infty$. But, the limit of Riemann sums is a definite integral. Thus,

Let $S$ be a solid that lies between $x = a$ and $x = b$. if the cross-sectional area of $S$ in the plane $P_x$, through $x$ and perpendicular to the $x$-axis, is $A(x)$, where $A$ is a continuous function, then the volume of $S$ is

When we use the volume formula $V = \int_a^b A(x) \, dx$, it is important to remember that $A(x)$ is the area of a moving cross section obtained by slicing through $x$ perpendicular to the $x$-axis.

Note: For a cylinder, the area of the cross section is constant: $\int x^*_i \Delta x$ for all $x$. In this case,
Example 7. Find the volume of the region $S$ whose base is a circular disk with radius $r$ and parallel cross sections perpendicular to the base are squares.

Example 8. Find the volume of the frustum of a pyramid with square base of side $b$, square top of side $a$, and height $h$. 