A function $f$ is called a __________________________ if it never takes on the same value twice; that is,

Graphically,

This gives the following geometric method for determining whether a function is one-to-one.

**Horizontal Line Test:** A function is one-to-one if and only if no horizontal line intersects its graph more than once.

**Example 1.** A function is given by a table of values. Determine whether it is one-to-one.

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>1.0</td>
<td>1.9</td>
<td>2.8</td>
<td>3.5</td>
<td>3.1</td>
<td>2.9</td>
</tr>
</tbody>
</table>
Example 2. A function is given by a graph. Determine whether it is one-to-one.

Example 3. Let $f(x) = 10 - 3x$. Determine whether $f$ is one-to-one.

Example 4. Let $g(x) = |x|$. Determine whether $f$ is one-to-one.
One-to-one functions are important because they are precisely the functions that possess inverse functions according to the following definition.

Let $f$ be a one-to-one function with domain $A$ and range $B$. Then its $f^{-1}$ has domain $B$ and range $A$ and is defined by

\[
\text{for any } y \text{ in } B.
\]

This definition says that if $f$ maps $x$ into $y$, then $f^{-1}$ maps $y$ back into $x$. (If $f$ were not one-to-one, then $f^{-1}$ would not be uniquely defined, and thus would not pass the Vertical Line Test.)

\[
\begin{align*}
\text{domain of } f^{-1} &= \text{range of } f \\
\text{range of } f^{-1} &= \text{domain of } f
\end{align*}
\]

Caution: Do not mistake the $-1$ in $f^{-1}$ for an exponent. That is,

The reciprocal $1/f(x)$ can be written as

Example 5. If $f(x) = x^5 + x^3 + x$, find $f^{-1}(3)$ and $f(f^{-1}(x))$. 

3
The letter $x$ is traditionally used as the independent variable, so when we concentrate on $f^{-1}$ rather than on $f$, we usually reverse the roles of $x$ and $y$ and write

We get the following ________________________________:

These cancellation equations say that $f^{-1}$ undoes what $f$ does and $f$ undoes what $f^{-1}$ does.

![Diagram](image)

**Question:** How do we compute inverse functions?

<table>
<thead>
<tr>
<th>How to Find the Inverse Function of a One-to-One Function $f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Write $y = f(x)$.</td>
</tr>
<tr>
<td>(2) Solve this equation for $x$ in terms of $y$ (if possible).</td>
</tr>
<tr>
<td>(3) To express $f^{-1}$ as a function of $x$, interchange $x$ and $y$. The resulting equation is $y = f^{-1}(x)$.</td>
</tr>
</tbody>
</table>

**Example 6.** Find a formula for the inverse of the function $f(x) = \frac{4x - 1}{2x + 3}$
The principle of exchanging $x$ and $y$ to find the inverse function also gives us the method for obtaining the graph of $f^{-1}$ from the graph of $f$:

- $f(a) = b$ if and only if $f^{-1}(b) = a$
- $\Rightarrow$ If the point $(a, b)$ is on the graph of $f$, then we get the point $(b, a)$ from $(a, b)$ by reflecting about the line $y = x$.

The graph of $f^{-1}$ is obtained by reflecting the graph of $f$ about the line $y = x$.

Example 7. Use the graph of $f$ to sketch the graph of $f^{-1}$.
The Calculus of Inverse Functions:

If $f$ is a one-to-one continuous function defined on an interval, then its inverse function $f^{-1}$ is also continuous.

“Proof:”

If $f$ is a one-to-one differentiable function with inverse function $f^{-1}$ and $f'(f^{-1}(x)) \neq 0$, then the inverse function is differentiable at $x$ and

“Proof:”

Note: If it is known in advance that $f^{-1}$ is differentiable, then its derivative can be computed more easily by using implicit differentiation.
Example 8. Let $f(x) = \sqrt{x-2}$.

a) Show that $f$ is one-to-one.

b) Find $(f^{-1})'(2)$ using the formula for the derivative of $f^{-1}$.

c) Calculate $f^{-1}(x)$ and state the domain and range of $f^{-1}$.

d) Calculate $(f^{-1})'(2)$ from the formula found in (c) and check that it agrees with the result of (b).

e) Sketch the graphs of $f$ and $f^{-1}$ on the same axes.
Example 9. Let \( f(x) = x^3 + 3 \sin x + 2 \cos x \). Find \( (f^{-1})'(2) \).

Example 10. If \( g \) is an increasing function such that \( g(2) = 8 \) and \( g'(2) = 5 \), calculate \( (g^{-1})'(8) \).
Example 11. Show that $h(x) = \sin x$, $x \in \mathbb{R}$, is not one-to-one, but its restriction $f(x) = \sin x$, $-\pi/2 \leq x \leq \pi/2$, is one-to-one. Compute the derivative of $f^{-1}(x) = \sin^{-1} x$. 