6.3 Logarithmic Functions

If \( a > 0 \) and \( a \neq 1 \), the exponential function \( f(x) = a^x \) is either strictly increasing or strictly decreasing and so it is one-to-one. It therefore has an inverse function \( f^{-1} \), which is called the \( \) and is denoted by \( \log_a x \). If we use the formulation of an inverse function,

then we have

Thus, if \( x > 0 \), then \( \log_a x \) is the exponent to which the base \( a \) must be raised to give \( x \).

**Example 1.** *Find the exact value of each expression.*

a) \( \log_{10} \sqrt{10} \)

b) \( \log_{1.5} 2.25 \)

The cancellation equations, when applied to \( f(x) = a^x \) and \( f^{-1}(x) = \log_a x \), become
**About the Logarithmic Function:**

- Has domain _______ and range _______
- It is continuous since it is the inverse of a continuous function.
- Its graph is the reflection of the graph of \( y = a^x \) about the line \( y = x \).
  
  ![Graph](image)

- Since \( \log_a 1 = 0 \), the graphs of all logarithmic functions pass through the point \((1, 0)\).

<table>
<thead>
<tr>
<th>If ( a &gt; 1 ), the function ( f(x) = \log_a x ) is a one-to-one, continuous, increasing function with domain ((0, \infty)) and range ( \mathbb{R} ). If ( x, y &gt; 0 ) and ( r ) is any real number, then</th>
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<tbody>
<tr>
<td>( 1 ) ( \log_a (xy) = )</td>
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<tr>
<td>( 2 ) ( \log_a \left( \frac{x}{y} \right) = )</td>
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<tr>
<td>( 3 ) ( \log_a (x^r) = )</td>
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These properties follow from the corresponding properties of exponential functions.

**Example 2.** *Find the exact value of \( \log_5 4 - \log_5 500 \).*
Example 3. Use the properties of logarithms to expand $\log_{10} \sqrt{\frac{x - 1}{x + 1}}$.

Example 4. Express $\log_{10} 4 + \log_{10} a - \frac{1}{3} \log_{10}(a + 1)$ as a single logarithm.

If $a > 1$, then

In particular, the $y$-axis is a vertical asymptote of the curve $y = \log_a x$.

Example 5. Find the limit: $\lim_{x \to 2^-} \log_5 (8x - x^4)$
Natural Logarithms

The logarithm with base $e$ is called the natural logarithm and has a special notation:

If we put $a = e$ and replace $\log_e x$ with $\ln x$, the defining properties of the natural logarithm become:

Example 6. Find the exact value of each expression.

a) $e^{-2\ln 5}$

b) $\ln(\ln e^{10})$

Logarithms with any base can be expressed in terms of the natural logarithm:

**Change of Base Formula:** For any positive number $a$ ($a \neq 1$), we have

Proof:
Example 7. Let $f(x) = \ln(x - 1) - 1$.

a) What are the domain and range of $f$?

b) What is the $x$-intercept of the graph of $f$?

c) Sketch the graph of $f$. 
Example 8. Solve each equation for $x$.

a) $\ln(x^2 - 1) = 3$

b) $e^{2x} - 3e^x + 2 = 0$

c) $e^{3x+1} = k$

d) $\log_2(mx) = c$
e) \( \frac{10}{1 + e^{-x}} = 3 \)

f) \( e^x = 10 \)

g) \( \ln(2x + 1) = 2 - \ln x \)
Example 9. Solve each inequality for $x$.

a) $1 < e^{3x-1} < 2$

b) $1 - 2 \ln x < 3$

Example 10. Find the domain of $f(x) = \ln x + \ln(2 - x)$. 
Example 11. Let $f(x) = \ln(2 + \ln x)$.

a) Find the domain and range of $f$.

b) Find $f^{-1}$.

c) Find the domain and range of $f^{-1}$.

Example 12. Find the inverse function.

a) $y = 2^{10x}$

b) $y = (\ln x)^2$, $x \geq 1$
Graph and Growth of the Natural Logarithm

Because the curve $y = e^x$ crosses the $y$-axis with a slope of 1, it follows that the reflected curve $y = \ln x$ crosses the $x$-axis with a slope of 1. The natural logarithm is a continuous, increasing function defined on $(0, \infty)$ and the $y$-axis is a vertical asymptote. Further, if we put $a = e$, we have:

Example 13. Find the limit: $\lim_{x \to \infty} [\ln(2 + x) - \ln(1 + x)]$
We have seen that $\ln x \to \infty$ as $x \to \infty$, but this happens very slowly. In fact, $\ln x$ grows more slowly than any positive power of $x$:

for any positive power $p$. So for large $x$, the values of $\ln x$ are very small compared with $x^p$.

**Example 14.** Make a rough sketch of the graph of $y = \ln(-x)$ and $y = \ln|x|$ by hand.