6.4 Derivatives of Logarithmic Functions

Goal: Find the derivatives of the logarithmic functions \( y = \log_a x \) and the exponential functions \( y = a^x \).

We will start with finding the derivative of \( y = \ln x \).

Example 1. Differentiate the function.

a) \( f(x) = \ln(\sin^2 x) \)

b) \( f(u) = \frac{u}{1 + \ln u} \)

c) \( f(x) = \ln |x| \)
The result of THE LAST EXAMPLE is worth remembering:

The corresponding integration formula is:

Example 2. Evaluate the integral.

a) \[ \int_{0}^{3} \frac{dx}{5x + 1} \]

b) \[ \int \frac{\sin(\ln x)}{x} \, dx \]
c) \[ \int \frac{e^x}{e^x + 1} \, dx \]

General Logarithmic and Exponential Functions

Recall: By the Change of Base Formula:

Thus, \( \frac{d}{dx} [\log_a x] \) is:

We also can now find the derivative of exponential functions \( y = a^x \):
Example 3. Differentiate the function.

\( a) \quad f(x) = \log_{5}(xe^{x}) \)

\( b) \quad g(x) = x \sin(2^{x}) \)

\( c) \quad y = \log_{2}(e^{-x} \cos \pi x) \)

\( d) \quad F(t) = 3^{\cos 2t} \)
Example 4. Find the domain of \( f(x) = \sqrt{2 + \ln x} \) and differentiate \( f \).

Example 5. Evaluate \( \int x^2 \cdot 2^x \, dx \)

Logarithmic Differentiation

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called ________________________________.

Steps in Logarithmic Differentiation:

1. Take natural logarithms of both sides of an equation \( y = f(x) \) and use the properties of logarithms to simplify.
2. Differentiate implicitly with respect to \( x \).
3. Solve the resulting equation for \( y' \).
Example 6. Use logarithmic differentiation to find the derivative of the function.

a) \( y = \frac{e^{-x} \cos^2 x}{x^2 + x + 1} \)

b) \( y = \sqrt{x}e^{x^2-x}(x + 1)^{2/3} \)
c) \( y = x^{\cos x} \)

d) \( y = (\sin x)^{\ln x} \)
The Number $e$ as a Limit

We have shown that if $f(x) = \ln x$, then $f'(x) = 1/x$. Thus, $f'(1) = 1$. We now use this fact to express the number $e$ as a limit.

Estimating numerically:

If we let $n = \frac{1}{x}$, then $n \to \infty$ as $x \to 0^+$. Thus,