Total: 182

College of the Canyons **Name:** Solutions
A. Morrow **Math 214 Exam 1 – Chapters 1-4** **Spring 2008**

This quiz is closed book. Answer the following questions NEATLY. Show all necessary work directly on the quiz. Answers without supporting work shown will receive no credit.
(6 points each part unless otherwise marked. Proofs 9 points each)

1) Fill in the Blanks with equivalent statements to make true properties. A, B, C invertible nxn matrices. k scalar. **u**, **v**, **w** $\in R^3$. *(1 point each)*

a) $(AB)^T = B^T A^T$
b) $(AB)^{-1} = B^{-1} A^{-1}$
c) $(A^T)^{-1} = (A^{-1})^T$
d) $(A + B)^T = A^T + B^T$
e) $\det(AB) = \det(A)\det(B)$
f) $\det(A^{-1}) = \frac{1}{\det(A)}$
g) $\det(kA) = k^n \det(A)$
h) $\det(A^T) = \det(A)$
i) $\|k\mathbf{u}\| = |k|\ \|\mathbf{u}\|$
j) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u})\cdot \mathbf{v} = \mathbf{u}\cdot(k\mathbf{v})$
k) $\mathbf{u} \times \mathbf{u} = 0$

2) Solve using Gaussian elimination: $\begin{cases} x_2 + 2x_3 + x_4 = 1 \\ x_1 - x_3 = 1 \\ 2x_1 + x_2 + x_4 = 3 \end{cases}$. Parametrize where appropriate. Write answer as a column vector.

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 2 & 1 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{-2R_1 + R_3} \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{-R_2 + R_3 \to R_3} \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = x_3 + 1$
$x_2 = 1 - 2x_3 - x_4$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \left\{ \begin{bmatrix} t+1 \\ 1-2t-s \\ t \\ s \end{bmatrix} \middle| \ t, s \in \mathbb{R} \right\}$$

/17

3) Consider the matrices below and find the following. If it is not possible, indicate why.

$$A=\begin{bmatrix}0&-1&2\\1&0&-1\end{bmatrix},\quad B=\begin{bmatrix}1&3&0\\0&2&-1\end{bmatrix},\quad C=\begin{bmatrix}2&0\\0&1\\-1&-2\end{bmatrix},\quad D=\begin{bmatrix}1&0&-1\\-1&1&1\\0&0&2\end{bmatrix}$$

a) tr(D)

$\boxed{4}$

b) A+B^T

Not Possible

A is 2x3

B^T is 3x2

c) det(CA)

$$CA=\begin{bmatrix}2&0\\0&1\\-1&-2\end{bmatrix}\begin{bmatrix}0&-1&2\\1&0&-1\end{bmatrix}$$

$$=\begin{bmatrix}0&-2&4\\1&0&-1\\-2&-1&0\end{bmatrix}$$

$\det(CA)=0+2(2)+4(-1)$

$=\boxed{0}$

(-4) if ||

d) (CA)^-1

$\det(CA)=0$

$\therefore$ CA not invertible

Correct Inverse of AC : (-2)

(-4) if not correct

30

e) D^-1

$$\left[\begin{array}{ccc|ccc}1&0&-1&1&0&0\\-1&1&1&0&1&0\\0&0&2&0&0&1\end{array}\right]$$

$\xrightarrow[\frac{1}{2}R_3\to R_3]{R_1+R_2\to R_2}$ $\begin{bmatrix}1&0&-1&1&0&0\\0&1&0&1&1&0\\0&0&1&0&0&\frac{1}{2}\end{bmatrix}$

$\xrightarrow{R_3+R_1\to R_1}$ $\begin{bmatrix}1&0&0&1&0&\frac{1}{2}\\0&1&0&1&1&0\\0&0&1&0&0&\frac{1}{2}\end{bmatrix}$

$$D^{-1}=\begin{bmatrix}1&0&\frac{1}{2}\\1&1&0\\0&0&\frac{1}{2}\end{bmatrix}$$

4) Prove: If A is a mxn matrix, then $(A^T)^T = A$. *(9 points)*

Given: A is m×n

Prove: $(A^T)^T = A$

Proof:

A m×n $\Rightarrow$ A^T n×m $\Rightarrow (A^T)^T$ is m×n

$\therefore$ A and $(A^T)^T$ have the same size

$$[(A^T)^T]_{ij} = [A^T]_{ji} = [A]_{ij}$$

$\therefore (A^T)^T = A$

Notation (2)

5) Consider the vectors $\mathbf{v} = (0, 1, 2)$ and $\mathbf{w} = (3, -1, -1)$. Find the following:

a) $||\mathbf{w}||$

$\sqrt{9+1+1}$

$\boxed{\sqrt{11}}$

b) $\mathbf{v}\cdot\mathbf{w}$

$0-1-2$

$\boxed{-3}$

c) $\text{proj}_{\mathbf{v}}\mathbf{w}$

$\frac{v\cdot w}{||v||^2}v$

$\frac{-3}{5}(0,1,2)$

$\boxed{\left(0, -\frac{3}{5}, -\frac{6}{5}\right)}$

$\text{proj}_w v$ (-4)

/27

6) Complete the definition: If $AB = BA = I$,
then A is invertible and B is an inverse of A. *(3 points)*

7) Prove that if A is invertible, then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$. *(9 points)*

Given: A invertible

Prove: A^T invertible & $(A^T)^{-1} = (A^{-1})^T$

Proof:

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I.$$

$\therefore$ A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

8) Given $\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = 5$, find $\begin{vmatrix} -2a_{21}+a_{11} & -2a_{22}+a_{12} & -2a_{23}+a_{13} & -2a_{24}+a_{14} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 3a_{41} & 3a_{42} & 3a_{43} & 3a_{44} \end{vmatrix}$.

$-3 \cdot 5$

-15

18

9) Let A be a square nxn matrix. Then A is invertible is equivalent to (make 6 equivalent statements): *(1 point each)*

a) $\det(A) \neq 0$

b) $RREF(A) = I$

c) $Ax = 0$ has only the trivial sol^{n}.

d) $Ax = b$ is consistent for all b

e) T_A is onto

f) T_A is 1-1

10) Let $A\mathbf{x} = \mathbf{o}$ be a homogeneous system of n linear equations in n unknowns, and let Q be an invertible nxn matrix. Show that if $A\mathbf{x} = \mathbf{o}$ has only the trivial solution, then $QA\mathbf{x} = \mathbf{o}$ has only the trivial solution. *(9 points)*

Given: $Ax = 0$ has only the trivial solution, Q invertible

Prove: $QAx = 0$ has only the trivial solution

Proof:

$Ax = 0$ has only the trivial solution

$\iff$ A is invertible.

$\therefore$ QA is invertible

$\therefore$ $QAx = 0$ has only the trivial sol^{n}.

/15

11) Prove: If **u** and **v** are vectors in 2-space and k is a scalar, then k(**u** · **v**) = (k**u**) · **v**. *(9 points)*

Given: $u = (u_1, u_2)$; $v = (v_1, v_2)$; $k \in \mathbb{R}$

Prove: $k(u \cdot v) = (ku) \cdot v$

Proof:

$$k(u \cdot v) = k(u_1 v_1 + u_2 v_2)$$
$$= (ku_1)v_1 + (ku_2)v_2$$
$$= (ku_1, ku_2) \cdot (v_1, v_2)$$
$$= (ku) \cdot v$$

missing parens once (−½)
more (−1)
• as (,) [illegible]

12) Complete the definition: Vectors **u** and **v** are orthogonal if, and only if, $u \cdot v = 0$.
(3 points)

13) Find 3 vectors orthogonal to **u** = (1, 2, 1, −1).

By Inspection

Vectors: $(0, 0, 1, 1)$
$(0, 0, 2, 2)$
$(0, 0, 3, 3)$

18

14) Complete the definition: A square matrix A is called symmetric if $A^T = A$.
(3 points)

15) Let A = [a_{ij}] be an nxn matrix. Determine whether A is symmetric when $a_{ij} = i - j$. Justify your answer.

Symmetric? Yes/No: NO

$$a_{ij} = i - j = -(j - i) = -a_{ji}$$

CounterEx: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is not symmetric

16) Prove: If $A^TA = A$, then A is symmetric and $A = A^2$. *(9 points)*

Given: $A^T A = A$

Prove: A is symmetric and $A = A^2$

Proof:

Symmetric:

$$A^T = (A^T A)^T = A^T A = A$$

$\therefore$ A is symmetric.

Prove $A = A^2$:

$$A^2 = A \cdot A = A^T A = A.$$

$\therefore A = A^2$

/18

17) Complete the following theorem statement. Properties of Linear Transformations. A transformation $T:R^n \to R^n$ is linear if and only if the following relationships hold for all vectors **u** and **v** in R^n and for every scalar c. *(3 points)*

a) $T(u+v) = T(u) + T(v)$ b) $T(cu) = cT(u)$

18) Using the theorem above, verify that T(x,y) = (y, y) is a linear operator. *(9 points)*

Given: $T(x,y) = (y,y)$; $u = (u_1, u_2)$; $v = (v_1, v_2) \in \mathbb{R}^2$; $c \in \mathbb{R}$

Prove: T linear

Proof:

$$T(u+v) = T(u_1+v_1, u_2+v_2) = (u_2+v_2, u_2+v_2)$$
$$= (u_2, u_2) + (v_2, v_2) = T(u) + T(v).$$

$$T(cu) = T(cu_1, cu_2) = (cu_2, cu_2) = c(u_2, u_2)$$
$$= cT(u).$$

$\therefore$ T is linear

19) Provide the standard matrix for T.

Standard Matrix: $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

18

20) Complete the definition. If $T:R^n \to R^n$ is a linear operator, then a scalar λ is called an eigenvalue of T if there is a nonzero x such that $T(x) = \lambda x$. *(3 pts)*

21) For T(x,y) = (y,y), find the following:

a) Characteristic equation

$$\det\begin{pmatrix} \lambda - 0 & -1 \\ -0 & \lambda - 1 \end{pmatrix} = 0$$

Characteristic Equation: $\lambda(\lambda - 1) = 0$

b) The eigenvalues

Eigenvalues: $0, 1$

c) Pick one eigenvalue. Find the eigenvectors corresponding to that eigenvalues.

Eigenvalue Picked: ______

$\lambda = 0$:

$$\left[\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & -1 & 0 \end{array}\right]$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_2 = 0$

$$\left\{\begin{bmatrix} t \\ 0 \end{bmatrix}\right\}$$

$\lambda = 1$:

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = x_2$

$$\left\{\begin{bmatrix} t \\ t \end{bmatrix}\right\}$$

Eigenvectors: ______

Bonus: Worth 3 extra points for a correct answer. No partial credit.
Find the eigenvalues for the derivative transformation from P1 to P1. If there are no eigenvalues, state such.

$T(ax+b) = a \rightarrow T(a,b) = (0,a)$

$[T] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $\quad \det\begin{pmatrix} \lambda & 0 \\ -1 & \lambda \end{pmatrix} = 0 \Rightarrow \lambda^2 = 0 \Rightarrow \boxed{\lambda = 0}$

/21