**College of the Canyons** **Name:** Solutions
**A. Morrow** **Math 214 Exam 3 – Chapters 7-8** **Spring 2008**

This exam is closed book. Answer the following questions NEATLY. Show all necessary work directly on the quiz. Answers without supporting work shown will receive no credit. *(Unless otherwise marked: 3 pts each definition, 6 each calculation, 9 each proof)*

1) Let $A = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$.

$$\det\begin{pmatrix} \lambda-3 & 0 & 1 \\ 0 & \lambda-2 & 0 \\ 1 & 0 & \lambda-3 \end{pmatrix} = 0$$

a) Find the characteristic equation for A: $(\lambda-2)^2(\lambda-4)=0$

b) Find the eigenvalues of A.

Eigenvalues: 2, 4

c) Find a basis for the eigenspace corresponding to each eigenvalue.

$\lambda = 2$:

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \downarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 = x_3$

Basis: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\lambda = 4$:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \downarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 = -x_3$, $x_2 = 0$

Basis: $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\vec{0}$ no e-vector (-3)

/18

1 CONTINUED) Let $A = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$.

d) Is A invertible? (Yes/No): Yes Why or why not? Explain using eigenvalues. *(3 pts)*

$\lambda = 0$ not an eigenvalue

e) Find the eigenvalues of $A^3$: $2^3 = 8$, $4^3 = 64$ *(2 points)*

f) Find a matrix P that **orthogonally** diagonalizes A.

Basis vectors already orthogonal.
Need to normalize them.

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \qquad \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

only vectors from (c) (-4)
is comment orth (+2)

P: $\begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$

g) Find the diagonal form of A corresponding to the matrix P that you found. *(3 points)*

$D = P^{-1}AP$ (+1)

Diagonal Form: $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

h) Find $A^n$ by writing out the appropriate matrix multiplication. Do not simplify fully. Complete answers will not have matrix exponentiation (including inverses).

$A^n$: $PD^nP^{-1} = PD^nP^T = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 4^n \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$

P not invertible /wrong inverse (-1)

wrong order (-1)

2) Precisely define **eigenvalue** and **eigenvector** of a nxn matrix. *(6 points)*

$\lambda$ is an eigenvalue of an nxn matrix A if $\exists\ \vec{x} \neq \vec{0}$ s.t. $A\vec{x} = \lambda\vec{x}$.
$\vec{x}$ is a corresponding eigenvector.

3) Complete the theorem statement. If A is an nxn matrix, then the following are equivalent.
a) A is diagonalizable.
b) A has n linearly independent eigenvectors.

4) Prove: If λ is an eigenvalue of A, **x** is a corresponding eigenvector, and s is a scalar then (λ – s) is an eigenvalue of A – sI.

Given: $\lambda$ is an e-val of A, x is corr. e-vector, $s \in \mathbb{R}$

Prove: $\lambda - s$ is an e-val of $A - sI$

Proof:

$$\begin{aligned}(A - sI)\vec{x} &= A\vec{x} - s\vec{x} \\ &= \lambda\vec{x} - s\vec{x} \\ &= (\lambda - s)\vec{x}\end{aligned}$$

$\Rightarrow$ $\lambda - s$ is an eigenvalue of $A - sI$

5) Complete the definition: If T: V → W is a function from a vector space V into a vector space W, then T is a **linear transformation** from V to W if, for all vectors **v**, **u** ∈ V and all scalars c,

a) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ and *(2 points)*

b) $T(c\vec{u}) = cT(\vec{u})$ *(2 points)*

6) Given $T:\mathbb{R}^2 \rightarrow \mathbb{R}^2$, $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ -x_2 \end{bmatrix}$.

a) Prove that T is a linear transformation.

Let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$ ; $c \in \mathbb{R}$

①
$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix}\right) = \begin{bmatrix} (x_1+y_1)+2(x_2+y_2) \\ -(x_2+y_2) \end{bmatrix}$$

$$= \begin{bmatrix} x_1+2x_2 \\ -x_2 \end{bmatrix} + \begin{bmatrix} y_1+2y_2 \\ -y_2 \end{bmatrix}$$

$$= T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + T\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right)$$

②
$$T\left(c\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}\right) = \begin{bmatrix} (cx_1)+2(cx_2) \\ -(cx_2) \end{bmatrix}$$

$$= c\begin{bmatrix} x_1+2x_2 \\ -x_2 \end{bmatrix}$$

$$= cT\left(\begin{bmatrix} x_1+2x_2 \\ -x_2 \end{bmatrix}\right)$$

6 CONTINUED) Given $T: R^2 \rightarrow R^2$, $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ -x_2 \end{bmatrix}$.

b) Find a basis for $R^2$ relative to which the matrix for T is diagonal.

Let B= std. basis.

$$[T]_B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \rightarrow (\lambda I - [T]_B) = \begin{bmatrix} \lambda-1 & -2 \\ 0 & \lambda+1 \end{bmatrix}$$

$\lambda = 1$:

$$\begin{bmatrix} 0 & -2 \\ 0 & 2 \end{bmatrix} \downarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$x_2 = 0$

Basis: $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

$\lambda = -1$:

$$\begin{bmatrix} -2 & -2 \\ 0 & 0 \end{bmatrix} \downarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$x_1 = -x_2$

Basis: $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

Basis: $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

c) Find the eigenvalues T: $1, -1$ *(2 points)*

7) Complete the statement of the Dimension Theorem for Linear Transformations: If T : V → W is a linear transformation from an n-dimensional vector space V to a vector space W, then

rank (T) + nullity(T) = n

8) Complete the definitions. If T : V → W is a linear transformation, then the

a) Kernel of T is

$\{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\}$

b) Range of T is

$\{T(\vec{v}) \mid \vec{v} \in V\}$

9) Prove ker(T) is a subspace of the domain of a linear transformation T.

Given: $T: V \rightarrow W$ linear

Prove: Ker (t) subspace of V

Proof:

Let $\vec{v}_1, \vec{v}_2 \in \ker(T)$, $c \in \mathbb{R}$.

$\therefore T(\vec{v}_1) = \vec{0}$, $T(\vec{v}_2) = \vec{0}$

① $T(\vec{0}) = \vec{0}$.

$\therefore \vec{0} \in \operatorname{Ker}(T)$

② $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) = \vec{0} + \vec{0} = \vec{0}$.

$\therefore \vec{v}_1 + \vec{v}_2 \in \ker(T)$

③ $T(c\vec{v}_1) = cT(\vec{v}_1) = c \cdot \vec{0} = \vec{0}$

$\therefore c\vec{v}_1 \in \ker(T)$.

$\therefore$ Ker (T) is a subspace of V

10) Prove if T: U → V is a one-to-one linear transformation and $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is linearly independent in U, then $\{T(\mathbf{u}_1), \ldots, T(\mathbf{u}_n)\}$ is linearly independent in V.

Given: $T: \mathcal{U} \to V$ 1-1, linear ; $\{\vec{u}_1, \ldots, \vec{u}_n\}$ LI

Prove: $\{T(\vec{u}_1), \ldots, T(\vec{u}_n)\}$ LI

Proof:

Let $c_1 T(\vec{u}_1) + \ldots + c_n T(\vec{u}_n) = \vec{0}$.

$\Rightarrow T(c_1\vec{u}_1 + \ldots + c_n\vec{u}_n) = \vec{0}$

$\Rightarrow c_1\vec{u}_1 + \ldots + c_n\vec{u}_n = 0$

$\Rightarrow c_1 = \ldots = c_n = 0$

$\Rightarrow \{T(\vec{u}_1), \ldots, T(\vec{u}_n)\}$ LI

11) Let T: $P_2 \to P_1$ by $T(a + bx + cx^2) = c + cx$.

a) Find matrix for T with respect to the bases $B = \{1, x, x^2\}$ and $B' = \{1 + x, x\}$.

$T(1) = 0 \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$T(x) = 0 \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$T(x^2) = 1 + x \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$[T]_{B',B}$ : $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

12) Let T: $P_2 \to P_1$ by $T(a + bx + cx^2) = c + cx$.

a) Find a basis for ker(T)

$$0 = c + cx$$

$$\Rightarrow c = 0.$$

$$\Rightarrow \ker(T) = \{a + bx\}$$

Basis: $\{1, x\}$

b) Find a basis for R(T)

$$R(T) = \{c + cx\}$$

$$= \{c(1+x)\}$$

Basis: $\{1 + x\}$

c) rank(T) = 1 *(1 point)*

d) nullity(T) = 2 *(1 point)*

13) Find an isomorphism between the vector space of all 3x3 diagonal matrices and $R^3$. *(3 points)*
*Note: You do not need to prove that it is an isomorphism.*

$$T\left(\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\right) = (a, b, c)$$

14) Determine whether the given linear transformations are one-to-one, onto, and/or bijective. Explain your choices.

a) T: R3 → R3, T(x, y, z) = (z, 0, y)

Circle all that apply: ~~1-1~~ ~~ONTO~~ ~~BIJECTIVE~~ [✓ NONE]

Why?

$\text{Ker}(T) = \{(x,0,0)\}$

$\neq \{\vec{0}\}$

$\Rightarrow$ not 1-1

$\text{rank}(T) = 3 - 1 = 2$

$\neq 3$

$\Rightarrow$ not onto

b) T: R3 → R2, T(x, y, z) = (z, y)

Circle all that apply: ~~1-1~~ [✓ ONTO] ~~BIJECTIVE~~ ~~NONE~~

Why?

$\text{Ker}(T) = \{(x,0,0)\}$

$\neq \{\vec{0}\}$

$\Rightarrow$ not 1-1

$\text{rank}(T) = 3-1=2$

$\Rightarrow$ onto

c) T: R2 → R3, T(x, y) = (x, 0, y)

Circle all that apply: [✓ 1-1] ~~ONTO~~ ~~BIJECTIVE~~ ~~NONE~~

Why?

$\text{Ker}(T) = \{(0,0)\}$

$\Rightarrow$ 1-1

$\text{rank}(T) = 2-0=2$

$\neq 3$

$\Rightarrow$ not onto

d) T: $P_1 \to P_1$, T(a + bx) = (a + b) + bx

Circle all that apply: [✓ 1-1] [✓ ONTO] [✓ BIJECTIVE] ~~NONE~~

Why?

$\text{Ker}(T) = \{0\}$

$\Rightarrow$ 1-1

$\text{rank}(T) = 2-0=2$

$\Rightarrow$ onto

**Bonus**: Worth 3 extra points for a correct answer. No partial credit. Write your answer on the back of this sheet.

15) Prove that eigenvalues are similarity invariant.

**Given:** $A = P^{-1}BP$, $\lambda$ is an e-val of $A$

**Prove:** $\lambda$ is an e-val of $B$.

**Proof:**

Let $\vec{x} \neq \vec{0}$ be an eigenvector of $A$ corresponding to $\lambda$.

$$\therefore A\vec{x} = \lambda\vec{x}$$

$$\Rightarrow P^{-1}BP\vec{x} = \lambda\vec{x}$$

$$\Rightarrow BP\vec{x} = P\lambda\vec{x}$$

$$\Rightarrow B(P\vec{x}) = \lambda(P\vec{x}).$$

Since $P$ is invertible and $\vec{x} \neq \vec{0}$, $P\vec{x} \neq 0$.

$\therefore \lambda$ is an eigenvalue of $B$.