College of the Canyons **Name:** Solutions
A. Morrow **Math 214 Test 2B – 1.5-2.4** **Spring 2010**

This quiz is closed book. Answer the following questions NEATLY. Show all necessary work directly on the quiz. Answers without supporting work shown will receive no credit.

Definitions *(3 points each unless otherwise marked)*

1) True or False: The following statements are equivalent to A is invertible. *(1 point each)*

   a) A is expressible as a product of elementary matrices. True

   b) The system $Ax = 0$ has only the trivial solution. False

   c) $\det(A) \cancel{=} 0$. False

2) For the statements that were false in #1, correct them. Write your corrections on the statements above. *(3 points)*

   See above

3) From 1.1-1.4:

   a) Complete the definition: A matrix A is an invertible matrix if ___.

   $\exists$ a matrix B s.t. $AB = BA = I$

4) True or False: If $\det(A) = \det(A^T)$, then A is symmetric. False

5) Which of the following are elementary matrices? Circle all that are. *(3 points)*

$$A = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \cancel{B} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \cancel{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

Procedures *(3 points each part). Show all work.*

6) Let $A = \begin{pmatrix} 1 & -4 \\ -3 & 2 \end{pmatrix}$. Find the eigenvalues of A.

$$\det\begin{pmatrix} \lambda-1 & 4 \\ 3 & \lambda-2 \end{pmatrix} = 0$$

$$(\lambda-1)(\lambda-2) - 12 = 0$$

$$\lambda^2 - 3\lambda - 10 = 0$$

$$(\lambda-5)(\lambda+2) = 0$$

$$\lambda = 5, -2$$

7) Let $A=\begin{pmatrix} 0 & 1 & -3 \\ 2 & 0 & 0 \\ 0 & 3 & -8 \end{pmatrix}$.

a) Find $\det(A)$ using any method.

$$= -2 \cdot \begin{vmatrix} 1 & -3 \\ 3 & -8 \end{vmatrix} = -2(1) = -2$$

b) Find $\operatorname{adj}(A)$

Matrix of Cofactors:

$$\begin{bmatrix} 0 & 16 & 6 \\ -1 & 0 & 0 \\ 0 & -6 & -2 \end{bmatrix}$$

Adjoint:

$$\begin{bmatrix} 0 & -1 & 0 \\ 16 & 0 & -6 \\ 6 & 0 & -2 \end{bmatrix}$$

c) Find A^{-1}, using adjoints.

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A) = \frac{1}{-2} \begin{bmatrix} 0 & -1 & 0 \\ 16 & 0 & -6 \\ 6 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -8 & 0 & 3 \\ -3 & 0 & 1 \end{bmatrix}$$

7) Let $A = \begin{pmatrix} 0 & 1 & -3 \\ 2 & 0 & 0 \\ 0 & 3 & -8 \end{pmatrix}$.

a) For A given above, solve $A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ for x_1 using Cramer's Rule.

$$\left.\begin{array}{l} \det(A) = -2 \\ \det\begin{pmatrix} 0 & 1 & -3 \\ 1 & 0 & 0 \\ 0 & 3 & -8 \end{pmatrix} = -1 \end{array}\right\} \Rightarrow x_1 = \frac{-1}{-2} = \frac{1}{2}$$

Proofs *(3 points each)*

8) Prove that A is invertible if and only if $A^T A$ is invertible.

$A^T A$ invertible

$\iff \det(A^T A) \neq 0$

$\iff \det(A^T)\det(A) \neq 0$

$\iff \det(A)\det(A) \neq 0$

$\iff \det(A) \neq 0$

$\iff A$ invertible

Prove: If $A^T A = A$, then A is symmetric.

Given: $A^T A = A$

Prove: A is symmetric

Proof:

$$A^T = (A^T A)^T = A^T (A^T)^T = A^T A = A$$

$\therefore$ A is symmetric

Bonus: *(3 extra points possible, no partial credit)*

Assume that A, B, and $A+B$ are all invertible. Show that $A^{-1}+B^{-1}$ is also invertible and find a formula for it.

Formula for inverse: $B(A+B)^{-1}A$

Proof:

First note that

$$A^{-1}(B+A)B^{-1} = A^{-1} + B^{-1}.$$

Therefore $A^{-1}+B^{-1}$ is expressible as a product of invertible matrices and is therefore invertible.

Moreover, $(A^{-1}+B^{-1})^{-1} = B(B+A)^{-1}A$