Math 214 Chapter 1
Notes and Homework

Systems of Linear Equations and Matrices

1.1: Introduction to Systems of Linear Equations

- A **linear equation** in n variables (unknowns) $x_1, x_2, \ldots, x_n$ is one that can be expressed in the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where $a_1, \ldots, a_n$ are constants.
  - What are some examples/non-examples?

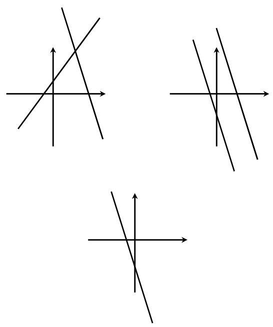
Solutions

- A **solution** of a linear equation $a_1x_1 + \ldots + a_nx_n = b$ is a sequence of n numbers $s_1, s_2, \ldots, s_n$ such that $a_1s_1 + \ldots + a_ns_n = b$
  - What is the difference between x_i and s_i?
- The set of all solutions is the **solution set** or **general solution**.
- **1.1 #3(b)**: Find the solution set of $3x_1 - 5x_2 + 4x_3 = 7$.

System of Linear Equations

- A **linear system** is a finite set of linear equations.
  - What is the general form of a linear system?
  - What is its corresponding **augmented matrix**?
- A sequence of numbers $s_1, \ldots, s_n$ is a **solution** of the system if it is a solution of every equation in the system.
  - How can you solve a system?
  - **Inconsistent** $\rightarrow$ no solution
  - **Consistent** $\rightarrow$ at least one solution

Systems: Types of Solutions



- One Solution
- No Solutions
- Infinitely Many Solutions

Elementary Row Operations

1. Multiply a row through by a nonzero constant
2. Interchange two rows
3. Add a multiple of one row to another row

- Equivalence to Operation on Equations
  1.
  2.
  3.

1.2: Gauss-Jordan Elimination

- Reduced Row-Echelon Form (RREF)
  1. The first nonzero entry from the left of a nonzero row is 1 (a **<u>leading one</u>** of the row)
  2. Rows of zeros are grouped at the bottom of the matrix
  3. For each nonzero row the leading one appears to the right and below any leading ones in preceeding rows.
  4. If a column contains a leading one, then all other entries in that column are zero

1	x	x	x	x	x
0	1	x	x	x	x
0	0	1	x	x	x
0	0	0	0	1	x
0	0	0	0	0	0

1	0	0	x	0	x
0	1	0	x	0	x
0	0	1	x	0	x
0	0	0	0	1	x
0	0	0	0	0	0

1	0	2	x	x	x
0	1	0	x	x	x
0	0	1	x	x	x
0	0	0	0	0	0
0	0	0	0	0	0

1.2 Example 4

- Gauss-Jordan Elimination to solve A**x** = **b**
  1. Form the augmented matrix [A | b]
  2. Obtain RREF using elementary row operations
  3. For each nonzero row of RREF, find solve the corresponding equation for the corresponding variable.

- **<u>Exercise</u>**: Solve by Gauss-Jordan Elimination
  – $x_1 + 3x_2 - 2x_3 + 2x_5 = 0$
  – $2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1$
  – $5x_3 + 10x_4 + 15x_6 = 5$
  – $2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6$

Existence of Solutions

- For what values of a, b, c, and d is the linear system
  - $ax + by = 0$
  - $cx + dy = 0$

  consistent?

Homogeneous Linear Systems

- A system of linear equations is **homogeneous** if the constant terms are all zero.
- Any homogeneous linear system always has the **trivial solution** ($x_1 = x_2 = \ldots = x_n = 0$)
  - Other solutions are called **nontrivial**
- When will there be nontrivial solutions for a homogeneous system?

1.3: Matrices and Matrix Operations

- A **matrix** is a rectangular array of numbers. The numbers in the array are the **entries**.
  - Row Matrix (row vector)
  - Column Matrix (column vector)
  - Square Matrix
  - Main Diagonal of a Matrix
- Notation: $A = [a_{ij}]$ or $[a_{ij}]_{mxn}$
  - m rows, n columns
  - a_{ij} = ij-th element (i^{th} row, j^{th} column)
  - What are m and n for a row matrix? Column matrix?
  - What are the entries along the main diagonal?

Matrix Operations

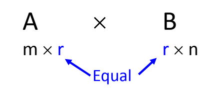
- Two matrices are **equal** iff they have the same size and corresponding entries are equal
  - A = B iff $a_{ij} = b_{ij}$ for all i and j
- Addition: $[A + B]_{ij} = a_{ij} + b_{ij}$
- Subtraction: $[A - B]_{ij} = a_{ij} - b_{ij}$
- Scalar Multiplication: $[cA]_{ij} = ca_{ij}$
- **Linear Combination**: $c_1A_1 + \ldots + c_nA_n$
- What do the definitions of these operations imply about the size of the matrices?

Matrix Multiplication

- If $A = [a_{ij}]$ is an m x r matrix and $B = [b_{ij}]$ is an r x n matrix, the product AB is an m x n matrix given by $AB = [c_{ij}]$ where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \ldots + a_{ir}b_{rj}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1r} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2r} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ir} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ b_{31} & b_{32} & \cdots & b_{3j} & \cdots & b_{3n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{in} \\ \vdots & \vdots & & & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mn} \end{bmatrix}$$

- Requirement:



Commutative?

2 More Operations: Transpose and Trace

- **Transpose**: A^T
  - Interchange rows and columns
  - $[A^T]_{ij} = [a_{ji}]$
  - Sometimes denoted A'
  - What are the dimensions of A^T ?
- **Trace**: tr(A)
  - Sum of entries on main diagonal
  - Undefined if A is not square
  - $tr(A) = \Sigma_i a_{ii}$

Practice

$$A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

- Find the following where possible:

1. A+B
2. $[A+B^T]_{21}$
3. $[A+B^T]_{13}$
4. AB
5. BA
6. tr(5C)
7. tr(A)
8. $[CA]_{21}$
9. AB^T

Matrix Multiplication as a Function

- **1.3 Example 10**: What is the effect of the matrix product **y** = A**x**?

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; x = \begin{bmatrix} a \\ b \end{bmatrix}$$

- **Ex**: Define the function **y** = A**x** that takes the point (x,y) and interchanges the coordinates.

1.4: Inverses; Rules of Matrix Arithmetic

- Properties of Matrix Arithmetic
  a) A + B = B + A
  b) A+(B+C) = (A+B)+C
  c) A(BC) = (AB)C
  d) A(B+C) = AB + AC
     - Verified in text
  e) (B+C)A = BA + CA
  f) A(B – C) = AB – AC
  g) (B – C)A = BA – CA
- Properties with Scalar Multiples
  h) a(B+C) = aB + aC
  i) a(B – C) = aB – aC
  j) (a + b)C = aC + bC
  k) (a – b)C = aC – bC
  l) a(bC) = (ab)C
  m) a(BC) = (aB)C = B(aC)
- To prove, must show
  - Same size and
  - Corresponding entries =

Zero Matrices

- A **zero matrix** is one whose entries are all zero.
- Properties of Zero Matrices
  - A + 0 = 0 + A = A
  - A – A = 0
  - 0 – A = -A
  - A0 = 0, 0A = 0

Identity Matrices

- Square matrices with 1's on the main diagonal and 0's off the main diagonal are **identity matrices**.
  - Notation: I_n = n x n identity matrix
- Main Property of Identity Matrices
  - $AI_n = A = I_mA$

The Inverse Matrix

- If a matrix B can be found such that AB = BA = I, then A is **invertible** and B is an **inverse** of A.
  - Are all matrices invertible?
  - Are all square matrices invertible?
- How do you find the inverse?

Properties of the Inverse Matrix

- **Thm 1.4.4**: If B and C are both inverses of A, then B = C.
  - *Prove…*
  - Hence, the notation A^{-1}
- **Thm 1.4.6**: If A and B are invertible matrices of the same size, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
  - *Prove…*

Properties of Transpose

- Properties of the Transpose
  a) $(A^T)^T = A$
  b) $(A + B)^T = A^T + B^T$
  c) $(kA)^T = kA^T$, k any scalar
  d) $(AB)^T = B^TA^T$
     - Verified in text

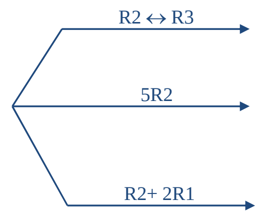
Inverse of the Transpose

- **<u>Thm 1.4.10</u>**: If A is an invertible matrix, then A^T is also invertible, and $(A^T)^{-1} = (A^{-1})^T$.
  - *Prove…*

1.5: Elementary Matrices and a Method for Finding A^{-1}

- An n x n matrix E is an **<u>elementary matrix</u>** if it can be obtained by performing a *single* elementary row operation on an identity matrix.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Row Operations by Matrix Multiplication

- **<u>Thm 1.5.1</u>**: Row Operations by Matrix Multiplication
  - If the elementary matrix E results for performing a certain row operation on I, then the product EA is the matrix that results from this same row operation on A.
- Verify by applying the elementary matrices from the last slide to a 3x3 matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Row Ops. and Inverse Row Ops.

Row Operation on I That Produces E	**Row Operation on E That Reproduces I**
Multiply row i by c ≠ 0	
Interchange rows i and j	
Add c times row i to row j	

- **<u>Thm 1.5.2</u>**: Every elementary matrix is invertible, and the inverse is also an elementary matrix.

Our First Equivalent Statements

- If A is an n x n matrix, then TFAE:
  a) A is invertible
  b) A**x** = **0** has only the trivial solution
  c) The RREF of A is I_n
  d) A is expressible as a product of elementary matrices
- Proof:
  – (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a)

In Practice: Finding the Inverse

- Given n x n matrix
  – Form augmented matrix: $[A \mid I_n]$
  – Transform A to RREF
  – Case 1: $[I_n \mid A^{-1}]$ or
  – Case 2: RREF is not I_n, and therefore A is not invertible

- **<u>EX</u>**: Find the inverse of

$$\begin{pmatrix} 2 & 1 & -1 \\ 6 & 4 & -1 \\ 4 & 2 & -3 \end{pmatrix}$$

1.6: Further Results on Systems of Equations and Invertibility

- **<u>Thm 1.6.1</u>**: Every system of linear equations has no solutions, exactly one solution, or infinitely many solutions.
  – Already know there is none, one, or more than one
  – Need to show: “more than one” implies infinite
  – Check that $x_1 + k(x_1 – x_2)$ is a solution for all scalars k if x_1 and x_2 are 2 solutions

Solving Systems with A^{-1}

- **Thm 1.6.2**: If A is an invertible n x n matrix, then for each n x 1 matrix **b**, the system of equations A**x** = **b** has exactly one solution, **x** = A^{-1}**b**.
  - *Prove…*
- **Ex**: Solve using inverses:

$$\begin{cases} x - y + 3z = 5 \\ 3x - y + 10z = 16 \\ 2x - 2y + 5z = 9 \end{cases}$$

- Some Octave Commands
  - A = [1 -1 3; 3 -1 10; 2 -2 5]
  - b = [5; 16; 9]
  - inv(A)*b
  - A2 = [1 -1 3 5; 3 -1 10 16; 2 -2 5 9]
  - rref(A2)

More Inverse Properties

- **Thm 1.6.3**: Let A be a square matrix.
  a) If B is a square matrix satisfying BA = I, then B = A^{-1}.
     - *Prove…*

  b) Similarly, if B is a square matrix satisfying AB = I, then B = A^{-1}.
- Consequence: Only need to show inverse on one side.

More Equivalent Statements

- If A is an n x n matrix, then TFAE:
  a) A is invertible
  b) A**x** = **0** has only the trivial solution
  c) The RREF of A is I_n
  d) A is expressible as a product of elementary matrices
  e) A**x** = **b** is consistent for every $n\times1$ matrix **b**
  f) A**x** = **b** has exactly one solution for every $n\times1$ matrix **b**
- Proof:
  - (a) $\Rightarrow$ (f) $\Rightarrow$ (e) $\Rightarrow$ (a)

A Fundamental Problem

- Find all $m\times1$ matrices **b** such that the system of equations A**x** = **b** is consistent.
- **1.6 Example 3**: What conditions must b_1, b_2, and b_3 satisfy in order for the following system of equations to be consistent?
  - $x_1 + x_2 + 2x_3 = b_1$
  - $x_1 + x_3 = b_2$
  - $2x_1 + x_2 + 3x_3 = b_3$

1.7: Diagonal, Triangular, and Symmetric Matrices

- **Diagonal Matrix**: A square matrix in which all entries off the main diagonal are zero
  - Octave: b is a column/row matrix.
    - diag(b) produces diagonal matrix with b on the diagonal
  - Is the zero matrix a diagonal matrix?
  - What does the inverse of a diagonal matrix look like (assuming it's invertible)?
  - When is a diagonal matrix not invertible?
  - Given a diagonal matrix D, how can you compute D^k?

Triangular Matrices

- **Lower triangular**: Entries *below* the main diagonal are nonzero
  - A square matrix A = $[a_{ij}]$ is lower triangular iff a_{ij} = 0 for _______ .
- **Upper triangular**: Entries *above* the main diagonal are nonzero
  - A square matrix A = $[a_{ij}]$ is upper triangular iff a_{ij} = 0 for _______ .
- **Thm 1.7.1**:
  - The transpose of a lower triangular matrix is upper triangular.
  - The product of lower triangular matrices is lower triangular.
    - *Proof in text…*
  - A triangular matrix is invertible iff its diagonal entries are all nonzero.
  - The inverse of an invertible lower triangular matrix is lower triangular.

Symmetric Matrices

- A matrix is **symmetric** if $A = A^T$.
  - Is a symmetric matrix necessarily square?
  - Entries are "mirror images" across diagonal
- **Thm 1.7.2**: If A and B are symmetric matrices of the same size and k is scalar:
  a) A^T is symmetric
  b) A + B and A – B are symmetric
  c) kA is symmetric
  - *Prove…*
- **Thm 1.7.3**: If A is an invertible symmetric matrix, then A^{-1} is symmetric.
  - *Prove…*

Homework

- Problems due on a given day will be announced in class
- 1.1: #1, 3(c), 4, 5(c), 6, 7, 8
- 1.2: #1, 4, 6, 8(d), 21, 25
- 1.3: #1, 3(d, f, k), 5(f, h), 7(c), 12(a), 13(a), 21, 25
- 1.4: #1(b), 3(d), 4(a,b), 6(a), 11, 13, 16, 17, 20(a), 21(a), 23, 24, 25, 26, 29(a)
- 1.5: #1, 2, 3(a, d), 5(b, c), 6(c), 7(e), 8(b), 14
- 1.6: #4, 11, 15, 16, 18, 23, 24, 26
- 1.7: #1(b, c), 3(b), 4(c, d), 5(a, b), 12, 15(a), 18, 20