Math 214 Chapter 2 Notes and Homework

Determinants

2.1: Determinants by Cofactor Expansion

- The **minor** M_{ij} of the entry a_{ij} is the determinant of the submatrix obtained from deleting the i^{th} row and the j^{th} column.
- The **cofactor** C_{ij} of the entry a_{ij} is given by $C_{ij} = (-1)^{i+j} M_{ij}$.
  - Signs of minors alternate over the matrix
  - $C_{ij} = M_{ij}$ if i + j is even
  - $C_{ij} = -M_{ij}$ if i + j is odd
- **Ex**: Find C_{23} and M_{23}:

$$\begin{pmatrix} 3 & -2 & 3 \\ 1 & 3 & 0 \\ 6 & -2 & 3 \end{pmatrix}$$

Evaluate a Determinant by Expanding by Cofactors

- $\det(A) = |A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}$
  - Cofactor Expansion along the i^{th} row
- $\det(A) = |A| = a_{1j}C_{1j} + a_{2j}C_{2j} + \dots + a_{nj}C_{nj}$
  - Cofactor Expansion along the j^{th} column
- **Tip**: Expand about the row or column with the most zeros
- NOT Efficient if n is large (use row reduction)
- **Ex**: Find the determinant:
  - Octave
  - A = [6 0 0; 2 -3 0; 7 -8 2]
  - det(A)

$$\begin{vmatrix} 6 & 0 & 0 \\ 2 & -3 & 0 \\ 7 & -8 & 2 \end{vmatrix}$$

Useful Lemma: Property with Cofactors

- Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then
  - $a_{i1}C_{k1} + a_{i2}C_{k2} + \dots + a_{in}C_{kn} = 0$ (for $i \neq k$)
    - Note: Cofactors are from a different row
  - $a_{1j}C_{1k} + a_{2j}C_{2k} + \dots + a_{nj}C_{nk} = 0$ (for $j \neq k$)
    - Note: Cofactors are from a different column
  - We won't prove this but the 3×3 case is verified in your book (page 88).
- **Thm**: $\det(A) \neq 0$ is equivalent to A is invertible
  - We'll prove this in section 2.3, but you can use it now

The Adjoint of a Matrix

- Adjoint of A = Transpose of matrix of cofactors

$$\begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ : & : & & : \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{pmatrix} \qquad \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ : & : & & : \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{pmatrix}^T$$

Matrix of Cofactors — Adj(A)

- **Thm 2.1.2** (Inverse of a Matrix Using Its Adjoint): If A is an invertible matrix, then

$$A^{-1} = (1/\det(A)) \operatorname{adj}(A).$$

  - *Prove by showing A adj(A) = det(A) I…*

Determinant of Triangular Matrix

- **Thm 2.1.3**: If A is an n×n triangular matrix (upper, lower, or both), then

$$\det(A) = a_{11}a_{22} \dots a_{nn}.$$

  - *Proof in text for n = 4.*

- **Thm 1.7.1(c)**: A triangular matrix is invertible iff its diagonal entries are all nonzero.
  - *Prove…*

Example: Finding Inverses

$$A^{-1} = \frac{1}{\det(A)} adj\ (A).$$

- This is a general formula for finding inverses

- Find adj(A) and A^{-1}: $A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$

- **Q**: Is the inverse of an upper triangular matrix always upper triangular? (Think back to chapter 1…)

Cramer's Rule

- **Cramer's Rule**: If A**x** = **b** is a system of n linear equations in n unknowns such that det(A) ≠ 0, then the system has a unique solution. This solution is:
  - $x_1 = \det(A_1)/\det(A), \dots , x_n = \det(A_n)/\det(A)$
  - A_i = matrix obtained by replacing the i^{th} column of A with the entries in the matrix **b**
  - *Prove…*

Using Cramer's Rule

- Given a system of n equations in n variables, the solution is (x_1, x_2, …, x_n) where
  - $x_1 = D_1/D$, $x_2 = D_2/D$, …, $x_n = D_n/D$
  - D = determinant of coefficient matrix
  - D_i = determinant of matrix formed by replacing the i^{th} column of coefficient matrix with the column of constants
- **Example**: Solve for x_1

$$\begin{cases} x_1 + 3x_2 = -2 \\ 2x_1 - 3x_2 + x_3 = 1 \\ 4x_1 + 5x_2 - 2x_3 = 0 \end{cases}$$

2.2: Evaluating Determinants by Row Reduction

- **Thm 2.2.1**: If a square matrix A has a row or column of zeros, then det(A) = 0.
  - *Prove…*
- **Thm 2.2.2**: Let A be a square matrix. Then det(A) = det(A^T).
  - *Proof in text (follows because expansion along row of A) is same as along corresponding column of A^T…*

Elementary Row Operations (2.2.3)

Row/Column Operation on A that produces B	Effect on Determinant:
Multiply row/column by constant k	det(B) = k det(A)
Interchange rows/columns	det(B) = – det(A)
Add multiple of row/column to another row/column	det(B) = det(A)

- **Thm 2.2.4**: Let E be an n×n elementary matrix. If E results from …
  a) multiplying a row of I_n by k, then det(E) = ______.
  b) interchanging two rows of I_n, then det(E) = ______.
  c) adding a multiple of one row of I_n to another, then det(E) = ______.

More on Row Ops. and Det

- **Thm 2.2.5**: If A is a square matrix with two proportional rows/columns, then det(A) = 0.
  - *Follows from last theorem…*
- Use Elementary Row Operations to Evaluate the Determinant
  - **Ex**: Evaluate determinant by first rewriting the matrix in triangular form:

$$\begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 3 & -4 & 5 \end{vmatrix}$$

2.3: Properties of the Determinant Function

- det(kA) = ____________
- **T/F**: det(A + B) = det(A) + det(B).
  - Can you think of a counterexample?
- **T/F**: det(AB) = det(A)det(B).
  - This would imply det (A^{-1}) = ____________. Why?
- **Q**: Under what conditions of A would det(A) = 0?

Determinant of Product: Lemma

- **Lemma 2.3.2**: If B is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then det(EB) = det(E) det(B).
  - Case 1: E results from multiplying I by k *(proof in text)*
  - *Cases 2 and 3 left to prove…*

Determinant of Product: Theorem

- **Lemma 2.3.2**: If B is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then det(EB) = det(E) det(B).
- **Thm 2.3.3**: A square matrix A is invertible iff det(A) ≠ 0.
  - USEFUL to show invertible!!! (Add to list of equivalences)
  - *Prove…*

Determinant of Product: Theorem

- **Thm 2.3.4**: If A and B are square matrices of the same size, then det(AB) = det(A) det(B).
  - *Case 1: A is not invertible (Apply Thm 2.3.3)*
  - *Case 2: A is invertible (Apply Lemma)*

Eigenvalues and Eigenvectors $Ax = \lambda x$

- Main Topic of Chapter 7
  - When is matrix multiplication the same as scalar multiplication?
  - $Ax = \lambda x \iff (\lambda I - A)x = 0$
- If $Ax = \lambda x$ has a nontrivial solution, then
  - λ = **eigenvalue**
  - x = **eigenvector**
- How to Find λ
  - **Characteristic Equation**: $\det(\lambda I - A) = 0$

Eigenvalues and Eigenvectors: Example $Ax = \lambda x$

- **Example 6**: Find the eigenvalues and corresponding eigenvectors for A.

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$$

2.4: A Combinatorial Approach to Determinants

- A **permutation** of a set of integers is n rearrangement of the integers
- **2.4 Example 1**:
  - How many permutations are there of the set {1, 2, 3}? List them all.
- In General
  - A permutation of {1, …, n} is denoted $(j_1, \ldots, j_n)$
    - Note: Sometimes denoted $j_1 j_2 \ldots j_n$
  - The set {1, 2, …, n} has _______ permuations.

Inversions

- An **inversion** occurs whenever a larger number precedes a smaller one in a permutation.
- Number of Inversions of $(j_1, \ldots, j_n)$
  - Sum (over i) the number of integers less than j_i that follow j_i
  - **Ex**: Find the number of inversions in (4, 2, 3, 1).
- The permutation is **even** if the number of inversions is even. Similarly, **odd**.
  - **Ex**: Determine the parity (even or odd) of each permutation of {1, 2, 3}.

Elementary Products

- Elementary Product
  - Any product of n entries from A, no two from the same row or column
  - General elementary product: $a_{1j_1}, a_{2j_2}, \ldots, a_{nj_n}$
- Signed Elementary Product
  - Elementary product multiplied by +1 if $(j_1, \ldots, j_n)$ is even and -1 if $(j_1, \ldots, j_n)$ is odd
  - **Ex**:
    - What is the sign on the term $a_{15}a_{24}a_{31}a_{43}a_{52}$?
    - $a_{15}a_{22}a_{31}a_{43}a_{54}$?

Determinant (Combinatorial Def.)

- Combinatorial Definition of Determinant
  - det(A) = sum of all signed elementary products from A
- Examples for general matrix $A = [a_{ij}]_{n\times n}$
  - n = 1
  - n = 2

Homework

- 2.1: #1, 2(a), 4, 7, 12, 15, 19, 22, 23, 25
- 2.2: #2, 3, 9, 11, 12, 15
- 2.3: #1(a), 2, 3, 4(a, c), 5(a, c, d, e), 7, 12(a), 13, 14(a), 15(for 14(a)), 16, 18
- 2.4: #1, 2, 3, 9, 17, 19