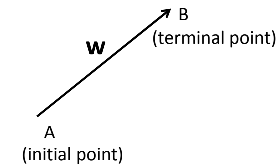
Math 214 Chapter 3 Notes and Homework

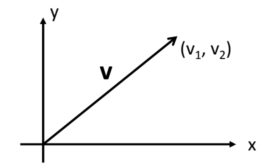
Vectors in 2-Space and 3-Space

3.1: Introduction to Vectors

- **w** = $\overrightarrow{AB}$

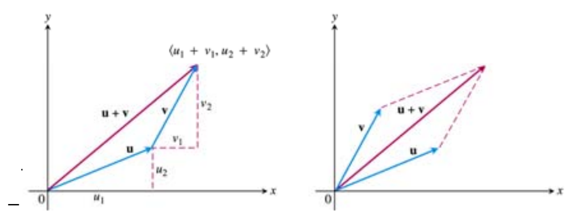


- Equivalent/Equal Vectors:
  - Same length and direction
    - **v** = **w**
    - $v_1 = w_1$ and $v_2 = w_2$
  - Every vector has an equivalent with initial point at the origin
    - Coordinates of terminal point: **components** of vector
- Written in Components:
  - **v** = (v_1, v_2)



Adding Vectors

- Geometric



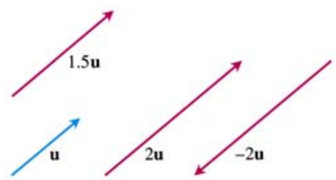
- By Components
  - **u** + **v** = $(u_1 + v_1, u_2 + v_2)$

Negatives and Zero Vectors

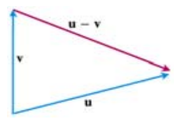
- **Zero vector**: The vector of length zero, denoted **0**.
  - **0** = (0, 0) in R^2
- Property: **0** + **v** = **v** + **0** = **v**
  - *Proof later…*
- **Negative vector of v**: -**v** is the vector that has the same magnitude of **v** but is oppositely directed.
- Property: **v** + (-**v**) = **0**
  - *Proof later…*

Scalar Multiplication

- Geometric



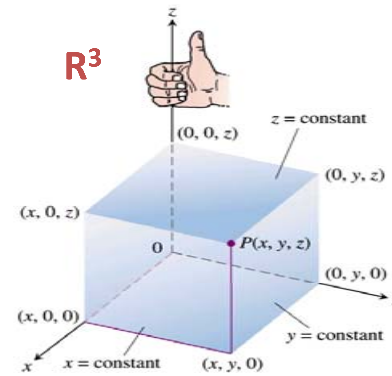
- Components
  - $k\mathbf{u} = (ku_1, ku_2)$
- Subtraction:
  - $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = (u_1 - v_1, u_2 - v_2)$



3-dimensional Coordinate Systems

- Three coordinate planes
  - xz-plane
  - yz-plane
  - xy-plane
- Divide space into eight **octants**
- x-coordinate = distance from the yz-plane, etc.

R^3



Exercises

- Let $P_1(-1, 3, 1)$ and $P_2(4, 7, 0)$. Find $\overrightarrow{P_1P_2}$.

- True/False (3.1 #21)
  1. If $\mathbf{x} + \mathbf{y} = \mathbf{x} + \mathbf{z}$, then $\mathbf{y} = \mathbf{x}$.
  2. If $\mathbf{u} + \mathbf{v} = \mathbf{0}$, then $a\mathbf{u} + b\mathbf{v} = \mathbf{0}$ for all a and b.
  3. Parallel vectors with the same length are equal.
  4. If $a\mathbf{x} = \mathbf{0}$, then either $a = 0$ or $\mathbf{x} = \mathbf{0}$.
  5. If $a\mathbf{u} + b\mathbf{v} = \mathbf{0}$, then **u** and **v** are parallel vectors.

3.2: Norm of a Vector; Vector Arithmetic

- **Properties of Vector Arithmetic**
  - **u**, **v**, **w** vectors; k, l scalars
  a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  b) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
     - *Proof in text…*
  c) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
  d) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
  e) $k(l\mathbf{u}) = (kl)\mathbf{u}$
  f) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
  g) $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$
  h) $1\mathbf{u} = \mathbf{u}$

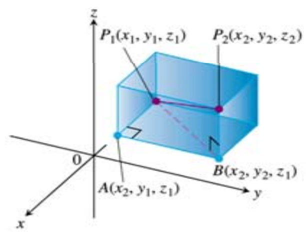
Tip: Be able to prove all of these using components (aka: analytically).

Distance in 3-Space

- In $\mathbf{R}^3$ the **distance** $|P_1P_2|$ between points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is given by

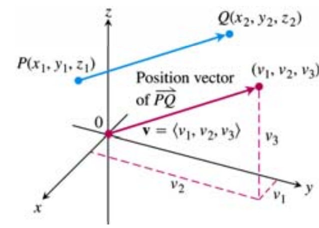
$$|P_1P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

- **Proof**: Apply the Pythagorean theorem twice



Norm (Length) of a Vector

- Note: $\overrightarrow{PQ} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$



- $||\mathbf{v}||$ = Norm of $\mathbf{v}$ = Length of $\mathbf{v}$
  = distance from (v_1, v_2, v_3) to $(0, 0, 0)$ = _________?
- **Unit Vector**: A vector of norm 1.
- **Property**: $||k\mathbf{u}|| = |k|\ ||\mathbf{u}||$

Unit Vectors

- **Unit vectors** have length 1

- Standard basis of $\mathbf{R}^3$
  - $\mathbf{i} = (1, 0, 0)$
  - $\mathbf{j} = (0, 1, 0)$
  - $\mathbf{k} = (0, 0, 1)$

- To make a vector $\mathbf{v}$ unit: $\mathbf{u} = \mathbf{v}/|\mathbf{v}|$
  - (If $\mathbf{v} \neq \mathbf{0}$)

The Triangle Inequality

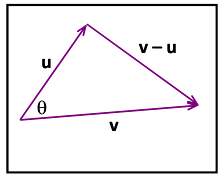
- The Triangle Inequality

$$||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$$

  - Why is it called the "triangle" inequality?
  - Is it possible to have equality?

3.3: Dot Product; Projections

- The **__Euclidean inner product__** or **__dot product__**:
  - If θ is the angle between the vectors **u** and **v**, then $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||\ ||\mathbf{v}|| \cos \theta$, or 0 if **u** or **v** is **0**.
  - Note: By angle between **u** and **v**, we mean $0 \le \theta \le \pi$
- In Components
  - $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$
  - Proof: Use Law of Cosines
  - $||\mathbf{v} - \mathbf{u}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 \ldots$
    $- 2||\mathbf{u}||\,||\mathbf{v}|| \cos \theta$.



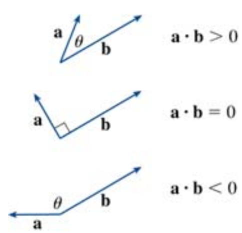
Euclidean Inner Product Example

- Use Both Versions of Euclidean inner product to find angles between two vectors
- **__Ex__**: Find the angle between **u** = (1, 0, -1) and **v** = (1, 1, 0).

Inner Product and Norm/Angles

- **__Thm 3.3.1__**: Let **u** and **v** be vectors.
  a) $\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2$
     - Corollary: $||\mathbf{v}|| = (\mathbf{v} \cdot \mathbf{v})^{1/2}$

  b) If **u** and **v** are nonzero and θ is the angle between them, then
     - θ is acute iff $\mathbf{u} \cdot \mathbf{v} > 0$
     - θ is obtuse iff $\mathbf{u} \cdot \mathbf{v} < 0$
     - $\theta = \pi/2$ iff $\mathbf{u} \cdot \mathbf{v} = 0$
- **u** and **v** are **__orthogonal__** iff $\mathbf{u} \cdot \mathbf{v} = 0$



Euclidean Inner Product Properties

- If **u**, **v**, and **w** are vectors and k is a scalar, then

a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (Symmetry)

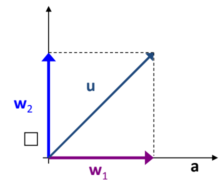
b) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (Additivity)

c) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$ (Homogeneity)
   - *Proof in text…*

d) $\mathbf{v} \cdot \mathbf{v} > 0$ if $\mathbf{v} \neq \mathbf{0}$, and $\mathbf{v} \cdot \mathbf{v} = 0$ if $\mathbf{v} = \mathbf{0}$ (Positive Definiteness)

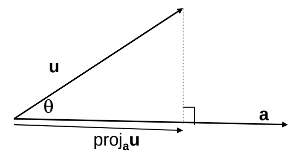
An Orthogonal Projection

- **<u>Problem</u>**: Decompose a vector **u** into the sum of two terms, one of which is parallel to a given vector **a** ($\mathbf{w}_1$) and the other perpendicular to **a** ($\mathbf{w}_2$)
  - $\mathbf{w}_1$ = the **<u>orthogonal projection of u onto a</u>** = **<u>vector component of u along a</u>** = $\text{proj}_{\mathbf{a}}\mathbf{u}$
  - $\mathbf{w}_2$ = the **<u>vector component of u orthogonal to a</u>** = $\mathbf{u} - \mathbf{w}_1$
  - **<u>Q</u>**: Is this possible? Why?
- **<u>Thm 3.3.3</u>**: If $\mathbf{a} \neq \mathbf{0}$, then $\text{proj}_{\mathbf{a}}\mathbf{u} = ((\mathbf{u} \cdot \mathbf{a}) / ||\mathbf{a}||^2)\, \mathbf{a}$
  - *Prove…*



Proj$_{\mathbf{a}}$**u** Examples

- **<u>3.3 Example 6</u>**: Let **u** = (2, -1, 3) and **a** = (4, -1, 2). Find the vector component of **u** along **a** and the vector component of **u** orthogonal to **a**.



- Alternative formulas
  - $||\text{proj}_{\mathbf{a}}\mathbf{u}|| = ||\mathbf{u}|| \cos\theta = |\mathbf{u} \cdot \mathbf{a}| / ||\mathbf{a}||$

3.3 Questions

- #27: What's wrong with each of the following:

1. $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$
2. $(\mathbf{u} \cdot \mathbf{v}) + \mathbf{w}$
3. $||\mathbf{u} \cdot \mathbf{v}||$
4. $k \cdot (\mathbf{u} + \mathbf{v})$

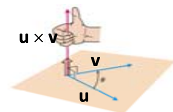
- #29: If $\mathbf{u} \neq \mathbf{0}$, is it valid to cancel **u** from both sides of the equation $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ and conclude that $\mathbf{v} = \mathbf{w}$?
- #31: Suppose that **u** and **v** are orthogonal. What famous theorem is described by $||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$?

3.4: The Cross Product

- Definition of Cross Product $\mathbf{u} \times \mathbf{v}$ for $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$
  - $\mathbf{u} \times \mathbf{v} = \langle u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1 \rangle$
  - Mnemonic: $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
- **<u>Questions</u>**:
  - What is $\mathbf{i} \times \mathbf{j}$? $\mathbf{j} \times \mathbf{k}$? $\mathbf{k} \times \mathbf{i}$?
  - What is $\mathbf{v} \times \mathbf{v}$?

Cross Product and Dot Product

- For **u**, **v**, and **w** in 3-space
  - a) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$
  - b) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$
    - Properties (a) and (b) automatically give you a vector orthogonal to **u** and/or **v**.
    - *Prove…*



Thm 3.4.2: Properties of Cross Product

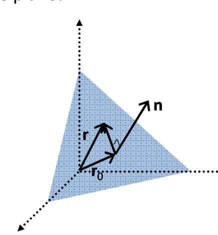
- Each is easily proven using components and definitions
  - Anticommutatitve: $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
  - Left/Right Distributive:
    - $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
    - $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$
  - Associative with Scalar: $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$
    - Warning: Not necessarily associative with cross product
    - $\mathbf{i} \times (\mathbf{j} \times \mathbf{j}) \neq (\mathbf{i} \times \mathbf{j}) \times \mathbf{j}$
  - $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
  - $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

Geometry and Cross Product

- If θ is the angle between **u** and **v** ($0 \leq \theta \leq \pi$), then
  $$||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}|| \; ||\mathbf{v}|| \sin \theta.$$
  - *Proof:* Examine $||\mathbf{u} \times \mathbf{v}||^2$

3.5: Lines and Planes in 3-Space

- A point and a normal vector are all that is needed to find the **equation of a plane**
  - Vector Equation: $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$
    - $\mathbf{n} = (a, b, c)$ and $\mathbf{r}_0 = (x_0, y_0, z_0)$ are fixed
    - $\mathbf{r} = (x, y, z)$ is variable
    - Important: **n** is **normal/perpendicular** to the plane!
  - Point-Normal Equation:
    - $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$
  - Linear (General) Equation:
    - $ax + by + cz + d = 0$
- Tip: To get the normal vector, cross 2 vectors that lie in the plane.



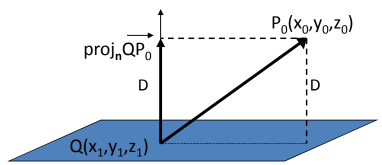
Plane Examples

- **Example**: Find an equation of a plane through the origin and the points (2, -4, 6) and (5, 1, 3)

- **Q**: In what ways can planes intersect?

Distance: Point to a Plane

- **Thm 3.5.2**: Find a formula for the distance D from a point $P_0(x_0, y_0, z_0)$ to the plane ax + by + cz + d = 0.



$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Homework

- 3.1: #1(b, j), 2(c, h), 3(c), 6(a, b), 9, 10, 15
- 3.2: #1(b, e, f), 2(c), 3(a, b, e, f), 4, 6, 9(a), 13, 14
- 3.3: #1(a, c), 2(a, c), 3(a, d), 4(a), 5(a), 6(c), 8, 9(d), 10, 12, 14, 16, 17, 18, 25
- 3.4: #1(a, b, c), 2(a), 6, 8(a), 15, 17(a), 20, 24(a), 33, 34
- 3.5: #1(a), 2(a), 4(a), 5(b), 7(a), 13(a), 18(b), 20, 33, 35, 38, 39(a), 40(a)