

Math 214 Chapter 4 Notes and Homework

Euclidean Vector Spaces

4.1: Euclidean n-space

- If n is a positive integer, then an **ordered n-tuple** is a sequence of n real numbers (a_1, \dots, a_n) .
- The set of all n -tuples is called **n-space** and is denoted \mathbb{R}^n .
- Operations on N-Tuples
 - $\mathbf{u} = (u_1, \dots, u_n), \mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n; k \in \mathbb{R}$
 - **Equality of n-Tuples:** $\mathbf{u} = \mathbf{v}$ if $u_1 = v_1, \dots, u_n = v_n$
 - **Definition of Addition:** $\mathbf{u} + \mathbf{v} = (u_1 + v_1, \dots, u_n + v_n)$
 - **Definition of Scalar Multiplication:** $k\mathbf{u} = (ku_1, \dots, ku_n)$

Properties of Vectors in n-Space

- **Properties of Vector Arithmetic**

– If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$; k, l scalars, then:

- a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- b) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- c) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$, where $\mathbf{0} = (0, \dots, 0)$
- d) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
- e) $k(l\mathbf{u}) = (kl)\mathbf{u}$
- f) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
- g) $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$
- h) $1\mathbf{u} = \mathbf{u}$

Note: These are the same properties we had in \mathbb{R}^2 and \mathbb{R}^3

Euclidean Inner Product

- Euclidean Inner Product
 - $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + \dots + u_nv_n$
- Euclidean Inner Product Properties
 - If \mathbf{u}, \mathbf{v} , and \mathbf{w} are vectors and k is a scalar, then
 - a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (Symmetry)
 - b) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (Additivity)
 - c) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$ (Homogeneity)
 - d) $\mathbf{v} \cdot \mathbf{v} \geq 0$, and $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if $\mathbf{v} = \mathbf{0}$ (Positive Definiteness)

Application of Inner Product: ISBN's

- 10 digit ISBN
 - 10th digit is a check
 - Let $\mathbf{a} = (1, 2, 3, 4, 5, 6, 7, 8, 9)$
 - $\mathbf{b} \in \mathbb{R}^n$
 - $\mathbf{a} \cdot \mathbf{b} \pmod{11} = 10^{\text{th}} \text{ digit}$
- **Ex:** Our Linear Algebra Book
 - ISBN: 0-471-66959-8

4.1: Example 2

- Simplify:
 - $(3\mathbf{u} + 2\mathbf{v}) \cdot (4\mathbf{u} + \mathbf{v})$
- Identify the properties used to simplify

Norm and Distance

- $\|\mathbf{u}\| = (\mathbf{u} \cdot \mathbf{u})^{1/2}$
- Cauchy-Schwarz Inequality in \mathbb{R}^n
 - $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\|$
- Properties of Length
 - a) $\|\mathbf{u}\| \geq 0$
 - b) $\|\mathbf{u}\| = 0$ iff $\mathbf{u} = \mathbf{0}$
 - c) $\|k\mathbf{u}\| = |k| \|\mathbf{u}\|$ (*Proof in text*)
 - d) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$
 - *Prove using Cauchy-Schwarz*

Orthogonality

- \mathbf{u} and \mathbf{v} are **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$
- Pythagorean Theorem in \mathbb{R}^n
 - If \mathbf{u} and \mathbf{v} are orthogonal, then
 - $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$

Alternative Forms of Vectors

- Write vectors as row or column vectors
- Matrix Form of Dot Product
 - If \mathbf{u} and \mathbf{v} are column vectors, then $\mathbf{u} \cdot \mathbf{v} = \mathbf{v}^T \mathbf{u}$
 - Also, if A is a matrix:
 - $(A\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (A^T \mathbf{v})$ AND $\mathbf{u} \cdot (A\mathbf{v}) = (A^T \mathbf{u}) \cdot \mathbf{v}$
 - These will be useful later on
 - Connect multiplication by a matrix to multiplication by its transpose

4.2: Linear Transformations from \mathbb{R}^n to \mathbb{R}^m

- General Function Vocabulary
 - A **function** is a rule f that associates each element in a set A one and only one element in a set B . $f: A \rightarrow B$ by $f(a) = b$
 - $A =$ Domain of f , $B =$ Codomain
 - $f(A) = \text{Im}(f) =$ Image of $f = \{b \in B \mid b = f(a)\} =$ Range of f
 - If $A = B = \mathbb{R}$, then f is **real-valued function of a real variable**
 - $f_1 = f_2$ if they have the same domain and $f_1(a) = f_2(a)$ for all a in the domain
- Example
 - Relationship: MEN \rightarrow WOMEN

Functions from \mathbb{R}^n to \mathbb{R}^m

- If the domain of f is \mathbb{R}^n and the codomain is \mathbb{R}^m , then f is a **transformation**
- If $m = n$, then $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is an **operator**.
- Notation for Transformations
 - $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$
 - $T(x_1, \dots, x_n) = (w_1, \dots, w_m)$
- Example
 - $T(x,y,z) = (x + 1, y - z)$

Linear Transformations

- If the equations of the transformation are linear, the transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a **linear transformation**.

$$w_1 = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n$$

$$w_2 = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n$$

$$w_m = a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n$$
- Linear transformations can be written in matrix form
 - $\mathbf{w} = A\mathbf{x}$
 - Special Notation: $T_A(\mathbf{x}) = A\mathbf{x}$
 - More Notation: $A = [T_A]$, the “standard matrix”
 - Goal for Now: Find the matrix A
- Example
 - $T(x,y,z) = (x + 1, y - z)$

Geometry Of Linear Transformations

- Zero Transformation
 - Page 184, Example 3
- Identity
 - Page 184, Example 4
- Reflection
 - Page 184, Tables 2 and 3 on page 185
- Projection
 - Pages 184, 186-187, Tables 4 and 5
- Rotation
 - Pages 186-189, Tables 6 and 7
- Dilation and Contraction
 - Pages 189-190, Tables 8 and 9

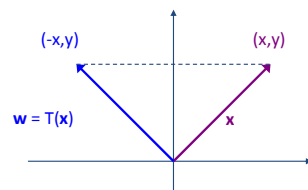
You are responsible for these transformations!

What Kind of Transformation?

- What kind of transformation is given by
 - $T(x,y,z) = (kx, ky, kz)$
 - What if $k > 1$?
 - What if $0 < k < 1$?
 - What if $k = 0$?
 - What if $k < 0$?
 - Is T injective? Surjective?

A Reflection Operator, $\mathbb{R}^2 \rightarrow \mathbb{R}^2$

- Consider the operator $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that maps each vector into its symmetric image about the y-axis
- What is T? Is T 1-1? Onto?

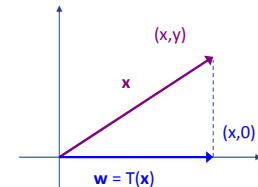


- What is $T(e_1), T(e_2)$?
- How does $[T(e_1) | T(e_2)]$ relate to the matrix you found?

A Projection Operator, $\mathbb{R}^2 \rightarrow \mathbb{R}^2$

- Consider the operator $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that maps each vector into its orthogonal projection on the x-axis

- What is T? 1-1? Onto?



- Note: Projection here could have been $T(x,y) = x \in \mathbb{R}$.

A Rotation Operator, $\mathbb{R}^2 \rightarrow \mathbb{R}^2$

- Consider the operator $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that rotates each vector through a fixed angle θ

$w = T(x) = (r \cos(\theta+\phi), r \sin(\theta+\phi))$

$(x, y) = (r \cos \phi, r \sin \phi)$

- Injective? Surjective?

Composition of Transformations

- $(T_B \circ T_A)(x) = T_B(T_A(x)) = T_{BA}(x)$
- Is composition commutative? Why or why not?
 - Consider reflection about y-axis followed by orthogonal projection to x-axis
 - Also consider reflection about the line $y=x$ followed by orthogonal projection to the x-axis

4.3: Properties of Linear Transformations from \mathbb{R}^n to \mathbb{R}^m

- The Defining Linearity Properties
 - A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **linear** if and only if the following hold for all $u, v \in \mathbb{R}^n$ and $c \in \mathbb{R}$
 - $T(u + v) = T(u) + T(v)$
 - $T(cu) = cT(u)$
- Ex:** Check that $T(x,y) = (-x, -y)$ is linear.

The Standard Matrix

- Thm 4.3.3:** Acquire the Standard Matrix
 - $[T] = [T(e_1) \mid T(e_2) \mid \dots \mid T(e_n)]$
 - Where e_i is the standard basis vector of \mathbb{R}^n with a 1 in the i^{th} position
- Consider the orthogonal projection $T_B: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ on the xy-Plane
 - Find $T(e_1)$, $T(e_2)$, and $T(e_3)$
 - What is B?

One-to-One Linear Transformations

- A linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **one-to-one (injective)** if T maps distinct vectors in \mathbb{R}^n into distinct vectors in \mathbb{R}^m
 - Example of one-to-one: Rotation
 - Nonexample: Projection
- More Equivalent Statements
 - a) A is invertible
 - b) The range of T_A is \mathbb{R}^n
 - c) T_A is one-to-one
 - These equivalent statements follow directly from our old equivalent statements
 - HUGE Consequence: Checking one-to-one means just checking if one number (the determinant) is nonzero

Inverse of a One-to-One Operator

- If T_A is 1-1, then T_A is invertible.
- Check that $T_{A^{-1}}$ is the inverse.
 - I.e: $(T_A \circ T_{A^{-1}})(\mathbf{x}) = (T_{A^{-1}} \circ T_A)(\mathbf{x}) = \mathbf{x}$
 - Notation: $[T^{-1}] = [T]^{-1}$
- 4.3 Example 4
 - Find the standard matrix for T^{-1} where $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the operator that rotates each vector in \mathbb{R}^2 through the angle θ

Geometric Interpretation of Eigenvectors

- Recall... Eigenvalues/eigenvectors of matrices
 - Definitions? How do you find them?
- For Linear Transformations
 - λ is an **eigenvalue of T** if there is a nonzero \mathbf{x} such that ____.
 - The **eigenvectors of T corresponding to λ** are ____.
 - How do the eigenvalues of T compare to the eigenvalues of its standard matrix A ?
 - $T(\mathbf{x}) = A\mathbf{x} = \lambda\mathbf{x}$ means that this linear transformation maps each eigenvector \mathbf{x} into ____?

Rotation: 4.3 Example 7

- Consider our rotation operator. What would you expect for eigenvalues?

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

- Find the characteristic equation and eigenvalues.

Projection: 4.3 Example 8

- Consider the orthogonal projection on the xy -plane

- What happens to vectors in the xy -plane under the mapping? Does this correspond to scalar multiplication?

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- What happens to vectors on the z -axis? Does this correspond to scalar multiplication?

- Find the eigenvalues and eigenvectors

4.4: Linear Transformations and Polynomials

- Polynomials and Vectors

- $p(x) = ax^2 + bx + c \in P^2$ corresponds to $(a, b, c) \in R^3$

- Use vectors to add polynomials:

- $p(x) = 3x^2 + 2x + 1$; $q(x) = 2x^2 - 3x + 2$

- Linear Operator: Derivative (Example 3)

- Show that $p'(x)$, where $p \in P^2$, is a linear operator, and find the standard matrix

Cryptography

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	0

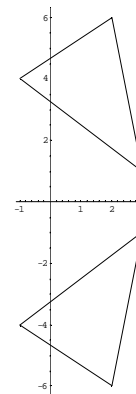
- Basic Hill Cyphers

- Write your message – Plaintext
 - Think up an **invertible** $n \times n$ matrix – A
 - Tricky Part: Must be invertible mod 26
 - Take your message n letters at a time (add dummy letters to fill it out if needed) – Plaintext Vectors
 - Apply T_A to the plaintext vectors – Cyphertext
 - Perform arithmetic (mod 26)

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Message:
20 5 17 27 15 30

Reviewing Transformations: Ex 1



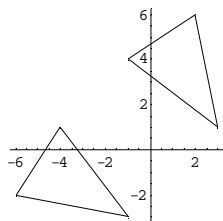
- A **reflection with respect to the x -axis** of a vector $u \in R^2$ is defined by the linear operator

$$L(u) = L([a_1 \ a_2]^T) = [a_1 \ -a_2]^T$$

- Find $[L]$

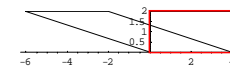
- Is L invertible? Injective? Surjective?

Reviewing Transformations: Ex 2



- A **reflection with respect to the line $y = -x$** of a vector $\mathbf{u} \in \mathbb{R}^2$ is defined by the linear operator $L(\mathbf{u}) = L([a_1 \ a_2]^T) = [-a_2 \ -a_1]^T$.
- Find $[L]$
- Is L invertible? Injective? Surjective?

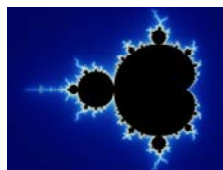
Reviewing Transformations: Ex 3



- A **shear in the $-x$ direction** of a vector $\mathbf{u} \in \mathbb{R}^2$ is defined by the linear operator $L(\mathbf{u}) = L([a_1 \ a_2]^T) = [a_1 + ka_2, a_2]^T$.
- Find $[L]$
- Is L invertible? Injective? Surjective?

Dynamical Systems

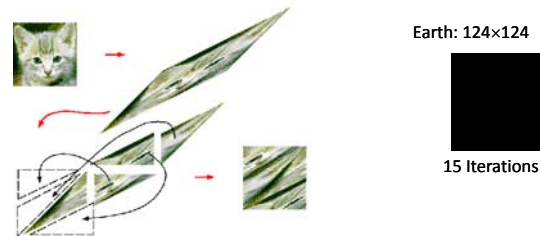
- Dynamical systems are systems that have a specific state or configuration at each point of time but that changes its state with time
 - Computer Graphics Programmer: Cool Pictures
 - Mathematician: iterated maps of the plane or algorithms that solve systems of differential equations
 - Physicist/Etc: Applications (“real-world” things such as planetary motion, the weather, the Stock Exchange, etc)



- Mandelbrot Set: Examines the fate of $x = 0$ under successive iterations of the map $f(x) = x^2 + c$ different values of c
 - <http://math.bu.edu/DYSYS/explorer/tour1.html>

Arnold's Cat Map

- $T(x,y) = (x + y, x + 2y) \pmod n$
- Apply to all points on an $n \times n$ picture



- Images Taken From:
 - <http://hypatia.math.uri.edu/~kulenm/diffegaturi/victor442/index.html>

Homework

- 4.1: #1(d), 4, 5(c), 6(f), 9(c), 11(a), 14(a), 15(b), 19, 20, 21, 22, 25, 26, 27(a), 28, 29, 30
- 4.2: #1(a,b), 2(b), 4(a), 5(a), 6(b), 8(a), 11(a), 12(a), 16(a,b), 18(a), 20(a), 22(a), 26
- 4.3: #1(a,c,e,g), 3, 4, 6(a), 7(a), 9(a,b), 12(a,c), 13(a), 18(a), 21, 22(a)
- 4.4: #1(b,c), 2(a), 4(a), 5(a), 7(a)
- 11.16: #1(a), 2, 3, 4
 - Note: If you did not purchase the applications version of the text, you will need to borrow the book from a friend or the TLC lab.