5.4: Basis and Dimension

- **Def**: If V is any vector space and S = {$\mathbf{v}_1$, …, $\mathbf{v}_n$} is a set of vectors in V, then S is a **basis** if the following hold:
  a) S is linearly independent
  b) S spans V
- Examples in R^2
  – {**i**, **j**} form a basis for R^2 (the standard basis)
  – S = {(1, 2), (3, 4)} also forms a basis for R^2

Bases in P_n

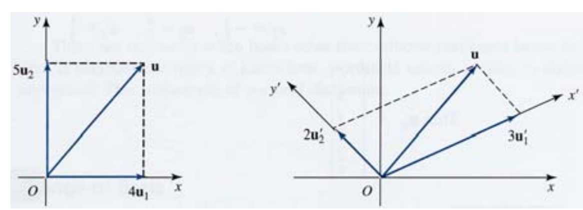
1. S = {1, x, …, x^n} form a basis for P_n (the standard basis)
2. S = {t^2 + 1, t – 1, 2t + 2} is a basis for P_2
   – *Verify for yourself…*
3. Find a basis for the subspace V of P_2, consisting of all vectors of the form
   – $ax^2 + bx + (a – b)$
   – Possible solution: {x^2 + 1, x – 1}
   – *Prove…*

Uniqueness of Basis Representation

- **Thm 5.4.1**: (Uniqueness of Basis Representation)
  If S = {$\mathbf{v}_1$, …, $\mathbf{v}_n$} is a basis for V, then every $\mathbf{v} \in V$ can be expressed in the form $\mathbf{v} = c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n$ in exactly one way
  – Proof by contradiction: Suppose there are 2 different ways
- Because it is unique, we have **coordinates** relative to S
  – $(\mathbf{v})_S = (c_1, \dots, c_n)$ – sometimes called the "coordinate vector"
- **Example**: What is the standard basis of M_{22}? Write coordinates of **v** = [a b; c d] in terms of the standard basis.

Visualizing the Coordinate Vector

- **Ex**: Find the coordinate vector of **v** = (4, 5) in R^2 in terms of
  a) S = The standard basis
  b) S' = {(2, 1), (-1, 1)}



Finite- vs. Infinite-Dimensional V.S.

- V is **finite-dimensional** if it has a finite basis.
  - **Ex**: P_n, R^n, M_{mn}
  - Finite-dimensional spaces are mostly what we study in math 214
- If no such basis exists, V is **infinite-dimensional**.
  - **Ex**: $F(-\infty, \infty)$, $C^\infty(-\infty, \infty)$, $C(-\infty, \infty)$
  - Consider $C^\infty(-\infty, \infty)$... What would a basis look like in terms of
    - Polynomials?
    - Trig functions?

Dimension

- **Thm 5.4.2**: Let V be finite dimensional and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be any basis.
  a) If a set has more than n vectors, then it is linearly dependent.
  b) If a set has fewer than n vectors, then it does not span V.
- **Thm 5.4.3**: All bases for a finite-dimensional vector space have the same number of vectors.
- **Def**: The **dimension** of V = dim(V) = number of vectors in a basis for V
  - Define the zero space to have dimension 0. (Basis for zero space = $\emptyset$)

Dimension Examples

- **5.4 Example 9**: Find the dimension of the following
  - R^n $\quad P_n \quad M_{mn}$
- **5.4 Example 10**: Determine a basis for and dimension of the solution space of
  - $2x_1 + 2x_2 - x_3 + x_5 = 0$
  - $-x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0$
  - $x_1 + x_2 - 2x_3 - x_5 = 0$
  - $x_3 + x_4 + x_5 = 0$
  - Solution found in 1.2: $x_1 = -s - t$, $x_2 = s$, $x_3 = -t$, $x_4 = 0$, $x_5 = t$

Dimension of Subspaces of M_{nn}

- **5.4 #32**: Find the dimension of the following:
  a) All diagonal n×n matrices
  b) All symmetric n×n matrices
  c) All upper triangular n×n matrices

A Fundamental Theorem

- **Thm 5.4.4**: (Plus/Minus Theorem)
  - Let $S \subseteq V$ be a nonempty set
  - a) If S is linearly independent and $\mathbf{v} \in V$ is outside of span(S), then $S \cup \{\mathbf{v}\}$ is linearly independent.
    - Example in R^3: {**i**, **j**}. **v** = **k**.
  - b) If $\mathbf{v} \in S$ is expressible as a linear combination of other vectors in S, then span(S – {**v**}) = span(S).
    - Example in R^3: {**i**, **j**, **k**, (1,1,1)}.

Consequence 1

- **Thm 5.4.5**: If V is n-dimensional, and if $S \subseteq V$ contains exactly n vectors, then S is a basis for V if either
  - S spans V *or*
  - S is linearly independent
- **5.4 Example 11**: By inspection, determine if the following are bases
  - a) { (-3, 7), (5, 5) } for R^2
  - b) { (2, 0, -1), (4, 0, 7), (-1, 1, 4)} for R^3

Consequence 2

- **Thm 5.4.6**: Let $S \subseteq V$ be a finite set
  - a) If S spans V but is not a basis, then S can be reduced to a basis by removing appropriate vectors.
    - One Way of Basis Construction (Continually delete and check for linear independence):
      - $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n = \mathbf{0}$.
      - Solve for $c_1, c_2, \dots, c_n$.
      - If they are all zero, this is the basis.
      - Otherwise delete ONE $\mathbf{v}_j$ with $c_j \neq 0$ and repeat the procedure.
  - b) If S is a linearly independent set that is not a basis, then S can be enlarged to a basis by inserting appropriate vectors.

Consequence 3

- **Thm 5.4.7**: If W is a subspace of a finite-dimensional vector space V, then
  - dim(W) ≤ dim(V); and
  - If dim(W) = dim(V), then W = V.
- **Factoid**: Any subspace of a finite-dimensional vector space is finite-dimensional.

5.5: Row Space, Column Space, and Nullspace

- Let A be an m×n matrix
  - **Row space** of A = subspace of R^n spanned by the row vectors of A
  - **Column space** of A = subspace of R^m spanned by the column vectors of A
  - **Nullspace** of A = solution space of A**x** = **0** (subspace of R^n)
- Questions of this (and the next) section
  - What relationships exist among them?

Example

- Find a basis for row space, column space, and null space of the following matrix:

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Recall: A**x** as a Linear Combination

- A = m×n matrix, **x** = n×1
- We can write A**x** as a linear combination of columns of A:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

- Important Later…
  - Column space
- Now: Can write A**x** = **b** as a linear combination of the columns of A

Column Space

- **Column space** of A = subspace of R^m spanned by the column vectors of A
- **Thm 5.5.1**: A**x** = **b** is consistent iff **b** is in the column space of A
  - ie: **b** = $x_1\mathbf{c}_1 + \ldots + x_n\mathbf{c}_n$, where $\mathbf{c}_i$ is the i^{th} column of A
  - Another View: **b** is in the range of T_A
- **5.5 Example 2**: Let A**x** = **b** as below. Show **b** is in the column space of A and express **b** as a linear combination of column vectors of A

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Row Equivalence & NS/CS

- **Thm 5.5.3/4**: Elementary row operations do not change the nullspace or row space of a matrix.
  - RS(A) = RS(R)
  - NS(A) = NS(R)
  - Note: This is **not** true for column space
  - Consider A = [1 3; 2 6] $\rightarrow$ B = [1 3; 0 0]
    - Row equivalent, but different column spaces
- **Thm 5.5.5**: If A and B are row equivalent, then
  - Column vectors of A linearly independent $\Leftrightarrow$ column vectors of B linearly independent
    - *Proof coming…*
  - **IMPORTANT**: A given set of column vectors of A forms a basis for the column space of A iff the corresponding column vectors of B form a basis for the column space of B.

Linear Independence of Column Vectors

- **Thm 5.5.5**: If A and B are row equivalent, then
  - Column vectors of A linearly independent $\Leftrightarrow$ column vectors of B linearly independent
    - *Proof outline…*

    1. Given A and B are row equivalent, what can we say about the solutions to A**x=0** and B**x=0**?
    2. What does this say about the linear combination of column vectors that solves each?
  - **IMPORTANT**: A given set of column vectors of A forms a basis for the column space of A iff the corresponding column vectors of B form a basis for the column space of B.

Bases for Row/Column Spaces

- **Thm 5.5.6**: If R is REF, then the row vectors with leading 1's (nonzero row vectors) form a basis for the row space of R, and the column vectors with leading 1's of the row vectors form a basis for the column space of R.
  - Focus: Leading 1's
- **5.5. Example 5**: Find bases for the row and column space of

$$\begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Non-RREF Form: Find RS(A) and CS(A)

- Find a basis for the row space and column space of A

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

Bases Using Row Operations

- **Useful**: To get a basis for a space spanned by vectors, use the row space of the matrix with those vectors as rows.
  - Note: The basis vectors found may not be **in** the original set
  - To Make Them From the Original Set:
    - Find Column Space of A^T, taking note of columns
    - Corresponding rows from A form basis
- **5.5 Example 8**: Find a basis for the row space of A consisting entirely of row vectors from A

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

Plan of attack?

Example 8 – RREF(A^T)

- **5.5 Example 8**: Find a basis for the row space of A consisting entirely of row vectors from A

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -2 & -5 & 5 & 6 \\ 0 & -3 & 15 & 18 \\ 0 & -2 & 10 & 8 \\ 3 & 6 & 0 & 6 \end{bmatrix} \longrightarrow \text{ref}(A^T) = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

► **Q**: When/why do you use A^T instead of A?

Finding Bases from S = {$\mathbf{v}_1$, …, $\mathbf{v}_n$}

- Steps to Find Bases for Span(S)
  1. Form A with $\mathbf{v}_i$ as columns
  2. R = REF(A)
  3. Columns with leading 1's in R correspond to columns in A → basis vectors for span(S)
  4. Express columns in R without leading 1's as linear combinations of preceding columns with 1's. These correspond to linear combinations of **v**'s.
- **5.5 Example 9**:
  - S = {(1, -2, 0, 3), (2, -5, -3, 6), (0, 1, 3, 0), (2, -1, 4, -7), (5, -8, 1, 2)}

  Find a basis for span(S) consisting entirely of vectors in S.

5.5 Example 9 (cont.)

- S = {(1, -2, 0, 3), (2, -5, -3, 6), (0, 1, 3, 0), (2, -1, 4, -7), (5, -8, 1, 2)}

Find a basis for span(S) consisting entirely of vectors in S

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix} \qquad ref(A) = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Summary: RS, NS, CS

- Let R = RREF(A)
- RS(A) = RS(R)
  - Row reduction does not change the row space
- NS(A) = NS(R)
  - Row reduction does not change nullspace
- Basis vectors of CS(A) CORRESPOND TO basis vectors of CS(R)
- **<u>Q</u>**: For what type of problems should you put vectors as rows? As columns?

5.5 #16: True/False, Justify

a) If E is an elementary matrix, then A and EA must have the same nullspace.

b) If E is an elementary matrix, then A and EA must have the same row space.

c) If E is an elementary matrix, then A and EA must have the same column space.

d) If A**x** = **b** does not have any solutions, then **b** is not in the column space of A.

e) The row space and nullspace of an invertible matrix are the same.

5.6: Rank and Nullity

- The Four Fundamental Matrix Spaces
  - Row Space of A
  - Column Space of A
  - Nullspace of A
  - Nullspace of A^T
  - Why don't we list row/column space of A^T?

dim(CS of A) = dim(RS of A)

- **Thm 5.6.1**: If A is any matrix, then the row space and column space of A have the same dimension.
  - R = REF(A)
  - Number Leading 1's in Columns = Number Nonzero Rows
- **Rank**(A) = dimension of row space of A
- **Nullity**(A) = dimension of nullspace of A
- **Thm 5.6.2**: If A is any matrix, then rank(A) = rank(A^T).

Rank/Nullity Example

- **5.6 Example 1**: Find the rank and nullity of the matrix

$$A=\begin{bmatrix}-1 & 2 & 0 & 4 & 5 & -3\\ 3 & -7 & 2 & 0 & 1 & 4\\ 2 & -5 & 2 & 4 & 6 & 1\\ 4 & -9 & 2 & -4 & -4 & 7\end{bmatrix}$$

Example 1: RREF(A)

$$\text{rref}(A)=\begin{bmatrix}1 & 0 & -4 & -28 & -37 & 13\\ 0 & 1 & -2 & -12 & -16 & 5\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

1. What is rank(A)?
2. What is nullity(A)?

The Dimension Theorem

- **Thm 5.6.3**: (Dimension Theorem for Matrices)
  If A is a matrix with n columns, then
  rank(A) + nullity(A) = n
- **Q**: Does the Dimension Theorem check on the last example?
- Using the Dimension Theorem
  - Find the rank first by row reducing
  - Find nullity by applying the Dimension Theorem
- **Factoid**: If A is m×n… Since dim(row space of A) = dim(column space of A),
  rank(A) ≤ min(m,n).

Adding On: The Equivalencies

b) A**x** = **0** has only the trivial solution

j) The column vectors of A are linearly independent

k) The row vectors of A are linearly independent

l) The column vectors of A span R^n

m) The row vectors of A span R^n

n) The column vectors of A form a basis for R^n

o) The row vectors of A form a basis for R^n

p) A has rank n

q) A has nullity 0

Homework

- 5.4: #1(c, d), 2, 3(c), 4(c), 6, 7(b), 10(b), 12, 17, 18(b), 19(b, c), 20(a), 23, 27(b)
- 5.5: #2(a), 3(a), 6(d), 7(a, c), 8(c), 9(c), 10(c), 11(a), 12(a), 13, 14, 15(b)
- 5.6: #2(a), 4(d), 5(b), 11

- Note: Yes/No answers not acceptable… All answers require justification.