Math 214 Chapter 7
Notes and Homework

Eigenvalues, Eigenvectors

7.1: Eigenvalues and Eigenvectors

- Review: A $\in M_{n\times n}$ and (nonzero) $\mathbf{x} \in R^n$
  - **x** is an **<u>eigenvector</u>** of A if $A\mathbf{x} = \lambda\mathbf{x}$
  - λ is an **<u>eigenvalue</u>**
    - Geometrically, what are eigenvalues/vectors?

  - **<u>Characteristic Polynomial</u>**: $\det(\lambda I - A)$
    - Why is this determinant called a "polynomial"?
    - What is the degree of this polynomial?
  - **<u>Characteristic Equation</u>**: $\det(\lambda I - A) = 0$
    - At most how many solutions can the characteristic equation have?
    - Why does solving this give us the eigenvalues?
  - **<u>Eigenspace</u>** of A corresponding to λ: $\{\mathbf{x} \mid A\mathbf{x} = \lambda\mathbf{x}\}$
    - How is this a null space problem?
    - Why do we know the eigenspace is a subspace?
    - Note: Separate eigenspace for each eigenvalue
- **<u>Example</u>**: Find the eigenvalues/vectors/spaces for I_n

Triangular Matrices

- **<u>Example 3</u>**: Find the eigenvalues of

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

- **<u>Thm 7.1.2</u>**:
  - In general, what can we say about eigenvalues of triangular matrices?

Powers of a Matrix

- **<u>Thm 7.1.3</u>**:
  - Given λ is an eigenvalue of A and **x** is a corresponding eigenvector, what can we say about eigenvalues and eigenvectors of A^2?
  - What can we say about eigenvalues and eigenvectors of A^k?
- **<u>Examples 5 and 6</u>**:
  - Find the eigenvalues and bases for the eigenspaces for A

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

  - Find the eigenvalues of A^7

Adding to Equivalent Statements

- **Thm 7.1.4**: A is invertible iff $\lambda = 0$ is not an eigenvalue of A.
  - Alternative Proof from Book:
    - Show the contrapositive
    - ie: $\lambda = 0$ is an eigenvalue iff A is not invertible
- **Example 7**: Based on our findings earlier, is A invertible?

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

7.2: Diagonalization

- The Matrix Diagonalization Problem
  - **Question 1**: Given a matrix A, does there exist a basis for R^n consisting of eigenvectors of A?
  - **Question 2**: Given a matrix A, does there exist an invertible matrix P such that $P^{-1}AP$ is diagonal?
    - A is called **diagonalizable** if there is an invertible matrix P such that $P^{-1}AP$ is diagonal. P **diagonalizes** A.
- **Thm 7.2.1**: If A is n×n, TFAE
  a) A is diagonalizable
  b) A has n linearly independent eigenvectors.
- Thm 7.2.1 establishes the equivalence of Questions 1 and 2

Proof of Theorem 7.2.1

- **Thm 7.2.1**: If A is n×n, TFAE
  a) A is diagonalizable
  b) A has n linearly independent eigenvectors.

Procedure for Diagonalizing

- Procedure (Suggested by Proof of 7.2.1)
  - Find roots of characteristic polynomial
    - Note: If roots are not real, A is not diagonalizable
  - For each eigenvalue, find a basis for the corresponding eigenspace
    - Note: If dim(eigenspace) < multiplicity of eigenvalue, A is not diagonalizable.
  - P = matrix whose columns are the n linearly independent eigenvectors.
  - $D = P^{-1}AP$
- **Example 1**: Diagonalize A, if possible

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \qquad \lambda = 2: \mathbf{p}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \lambda = 1: \mathbf{p}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Computing Powers of a Matrix

- Main Application to Diagonalization
  - Suppose A is diagonalizable
  - $D = P^{-1}AP$
  - A^k = ______________
- **Example 6**: Find A^{13}

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}, P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

A Non-Example

- Determine if A is diagonalizable.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Some of the Theory

- **Thm 7.2.2**: If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent.
  - *Proof by induction on k*
- **Thm 7.2.3**: If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.
- Note: The eigenvalues are in the same positions on the diagonal matrix as the corresponding eigenvectors on P.

Fibonacci: A Non-Recursive Formula

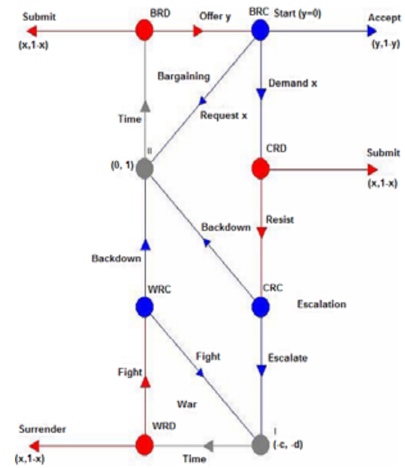
- Fibonacci Sequence: 1, 1, 2, 3, 5, 8, …
  - $a_1 = a_2 = 1$
  - What is the recursive formula for a_n?
- Find a matrix, A, that you can multiply to $[a_{n+1}\ a_n]^T$ to get $[a_{n+2}\ a_{n+1}]^T$.
- **HW**: Find the Non-Recursive Formula
  1. Find the eigenvalues and corresponding eigenvectors for A
     - Note: Your answer will contain square roots of 5
  2. Diagonalize A
  3. Use the diagonal form to find A^n;
  4. Use A^n to find the 21st term in the Fibonacci sequence.
     - You should get 10946

Math 214 Markov Chains and Game Theory

Repeated Games

Bargaining: Game Theory

- Defender has appropriate prize
- Challenger is demanding a share under threat of costly action
- These games can be solved using Markov Chains



http://userwww.sfsu.edu/~langlois/Appendix%20Two.pdf

Markov Chains

- A **Markov chain** is a process in which the probability of the system being in a particular state at a given time depends only on its state at the immediately preceding time.
- States on Preceding Example
  - Status Quo II (SQ), War (WR), Submit (SB)
- Player moves in game theory are characterized by probabilities, so there are probabilities associated with being in any state, starting from a given state

Transition Matrix

Preceding State

	SQ	WR	SB	
SQ	___	___	___	
WR	___	___	___	New State
SB	___	___	___	

- Let $P = [p_{ij}]$ = Transition Matrix of a Markov Chain.

  Then
  - p_{ij} = Probability of arriving in state i from state j
  - $0 \le p_{ij} \le 1$
  - Each column sums to 1
- Unfortunately, solving games requires a bit more probability than is required for this course.
  - Instead, we'll take a closer look at Markov chains

State Vector

- If P is a transition matrix, $x^{(n)}$ is the state vector at n^{th} observation
  - $x^{(n+1)} = Px^{(n)}$
- Initial State Vector: $x^{(0)}$
  - State vector at the beginning of time
- Hence (Thm 11.6.1)
  - $x^{(n)} = Px^{(n-1)} = P^2x^{(n-2)} = \ldots = P^nx^{(0)}$
  - We will apply methods of 7.2 to find P^n.

Experiment

- Write down which computer you used most recently and which computer you used before that.
- There are only three computers and nine choices:
  - H – (at home, your own PC or lap-top),
  - C - (one of the computers on-campus) or
  - O – (other).

Recent \ Past	H	C	O
H			
C			
O			

► **<u>Ex</u>**: Find the transition matrix and the initial state vector.

Steady-State Vector

- The Big Question
  - What happens as the number of observations increases? Does the Markov chain approach a resting point or **<u>equilibrium</u>**?
- **<u>Steady-State Vector</u>**
  - Vector **q** such that P**q** = **q**
  - **<u>Q1</u>**: If a Markov chain has a steady-state vector, what eigenvalue does P have?
  - **<u>Q2</u>**: How can we find **q** (there are 2 ways)?
    - Keep in mind that **q** must be a probability vector (ie: entries are nonnegative and sum to 1)

Steady-State: An Example

- Customers shop either at A or B
  - If a customer shops at A one day, then the customer will shop at A the next day with probability 4/5.
  - If a customer shops at B one day, then the customer will shop at B the next day with probability 1/3.

1. Find the steady-state vector.
2. Over the long term, what are the chances that a customer shops at A on any given day?

7.3: Orthogonal Diagonalization

- Orthogonal Diagonalization Problem
  - Given a matrix A
  - Q1: Does there exist an orthonormal basis for R^n under the Euclidean inner product that consists of eigenvectors of A?
  - Q2: Does there exist an orthogonal matrix P such that $P^{-1}AP = P^TAP$ is diagonal?
    - If yes, then A is called **orthogonally diagonalizable**.
- **Thm 7.3.1**: If A is n×n, TFAE
  a) A is orthogonally diagonalizable
  b) A has an orthonormal set of n eigenvectors.
  c) A is symmetric

Orthogonal Diagonalizability and Symmetric Matrices

- **Thm 7.3.2**: If A is symmetric, then
  a) The eigenvalues of A are real numbers
  b) Eigenvectors from different eigenspaces are orthogonal.
- Procedure for Orthogonally Diagonalizing
  1. Find basis for each eigenspace
  2. Apply Gram-Schmidt
  3. Form P

Orthogonal Diagonalization: Example

$$A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$$

- $\lambda_1 = -2$, $\lambda_2 = -2$, $\lambda_3 = 4$
- $\mathbf{u}_1 = [-1\ 1\ 0]^T$; $\mathbf{u}_2 = [-1\ 0\ 1]^T$; $\mathbf{u}_3 = [1\ 1\ 1]^T$
- Apply Gram-Schmidt

Homework

- 7.1: #1(d), 2(d), 3(d), 4(a), 5(a), 6(a), 7(a), 8(a), 10(c), 12, 13(b,c), 20, 21, 22(a,b)
- 7.2: #1, 2, 5, 9, 11, 13, 15, 16, 19
- HW #1-4 From Slide "Fibonacci: A Non-Recursive Formula"
- 11.6: #1, 3(c), 4(b – find P^n and find the limit as $n \to \infty$), 7, 8
  - Disregard need to show regular on all problems
- 7.3: #1(a, c, d, e), 3, 5, 6, 7, 11