## Math 214 Final Review Exercises

### Vector Spaces

1. Define a vector space.

2. Define a subspace.

3. What do you need to verify in order to prove a set is a subspace of a given vector space?

4. Provide an example of a subspace of $R^2$, and prove that it is.

5. Provide an example of a subspace of $P_3$, and prove that it is.

6. Provide an example of a subspace of $M_{nn}$, and prove that it is.

7. Determine if the following are subspaces of $M_{22}$ or not. Justify your answers with proofs (if they are) or counterexamples (if they aren't). You may quote any result from the text or the lecture notes.

    a. W = {A | A is diagonal}

    b. W = {A | A has integer entries}

8. All subspaces contain at least one vector. What vector is it?

9. Provide an example of a subset of $R^2$ that is closed under scalar multiplication, but not addition.

10. Provide an example of a subset of $R^2$ that is closed under addition, but not scalar multiplication.

11. Define linear independent set.

12. Define span of a set.

13. Define basis of a vector space.

14. Provide an example of a linear independent set in $P_3$ that does not span $P_3$. Prove that the set you found is linearly independent.

15. Provide an example of a set that spans $P_3$ that is not linearly independent. Prove that the set you found spans $P_3$.

16. Provide an example of a basis for $P_3$ that is not the standard basis. Prove that the set you found is a basis.

17. True/False (and provide reasons): If $\mathbf{v}_1, \ldots, \mathbf{v}_4$ are linearly independent and $\mathbf{v}_2, \ldots, \mathbf{v}_5$ are linearly independent, then the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_5$ are linearly independent.

18. Find b such that (-1, b, 2, 3) is in the span of (1, 2, 3, 4) and (3, 4, 4, 5).

19. Provide a matrix A for yourself. For this matrix, find bases for the nullspace, column space, and row space. Find the rank and nullity of the matrix.

20. State the Dimension Theorem for matrices.

21. A is invertible is equivalent to _________________. Fill in that blank with at least 10 other statements.

22. Let $A = \begin{bmatrix} 1 & a & 0 \\ a & 1 & a \\ 0 & a & 1 \end{bmatrix}$. For what values of a is A not invertible?

23. Find examples of $n \times n$ matrices A and B such that A, B are not invertible but A + B is.

24. True/False. Give a reason if true or a counterexample if false.

   a. If $A^T$ is invertible, then A is invertible.

   b. If AB = **0**, then either A = **0** or B = **0**.

25. For the matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$,

   a. Find the inverse

   b. Find the determinant

## Inner Product Spaces

26. Define an inner product on a vector space.

27. Provide an example of a function that is an inner product on $P_1$. Prove that it is an inner product.

28. Provide an example of a function that is an inner product on $C(-\infty,\infty)$. Prove that it is an inner product.

29. Define orthogonal vectors.

30. For your inner product above on $P_1$, provide an example of two vectors that are orthogonal.

31. For your inner product above on $P_1$, provide an example of two vectors that are not orthogonal.

32. For your inner product above on $P_1$, find $||1||$.

33. For your inner product above on $P_1$, find the angle between 1 and 1 + x.

34. For your inner product above on $P_1$, find $\text{proj}_1(1+x)$.

35. For your inner product above on $P_1$, use the Gram-Schmidt process to construct an orthonormal basis from the basis {1, 1 + x}.

36. Let W be the span of three vectors $\mathbf{v}_1 = (1, -1, 3, -2)$, $\mathbf{v}_2 = (1, 9, 1, -10)$, and $\mathbf{v}_3 = 2\mathbf{v}_1 - \mathbf{v}_2$ in $R^4$. What is the dimension of W? Find an orthonormal basis for W.

37. Let **u**, **v** be orthogonal to each other in an inner product space. Show that $||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$.

38. Let A be an $m \times n$ matrix. Prove that NS(A) is orthogonal to RS(A).

39. Define orthogonal matrix.

40. Show that a matrix that is both orthogonal and lower triangular must be diagonal.

## Eigenvalues, Eigenvectors

41. For a given matrix A, define eigenvalues and eigenvectors.

42. What conditions must be met in order to diagonalize a matrix?

43. Find eigenvalues, eigenvectors, eigenspaces for the following. Diagonalize the matrices where possible:

   a. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$

44. For a matrix A, what are the eigenvalues and eigenvectors of $A^n$?

45. Without performing any computations, how can you look at a matrix and know if it is orthogonally diagonalizable?

46. Orthogonally Diagonalize $A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{bmatrix}$. Find det(A).

47. By looking only at the eigenvalues, how can you determine if A is invertible?

## Linear Transformations

48. Define linear transformation.

49. Define kernel of a transformation and range of a transformation.

50. Provide a formula for a transformation $T:R^3 \rightarrow R^2$, and prove that it is linear.

    a. Find a basis for ker(T).

    b. Without finding R(T), find rank(T) and nullity(T).

    c. Find a basis for R(T).

    d. Is the transformation you created onto/surjective? 1-1/injective? Bijective? Could you have created an isomorphism?

51. Provide a transformation $T: P_1 \rightarrow P_1$, and prove that it is linear.

    a. Find an orthonormal basis relative to which the matrix for T is diagonal.

    b. Is the transformation you created onto/surjective? 1-1/injective? Bijective? Could you have created an isomorphism?

52. Prove that eigenvalues are similarity invariant.

53. State the dimension theorem for linear transformations. Verify that it holds on the transformations you created above.

54. Prove that if B is invertible then AB is similar to BA.