Multiplying and Dividing Fractions
Overview

- Fractions and Mixed Numbers
- Factors and Prime Factorization
- Simplest Form of a Fraction
- Multiplying Fractions and Mixed Numbers
- Dividing Fractions and Mixed Numbers
Intro to Fractions and Mixed Numbers
What is a Fraction?

- Has two parts:
  - Numerator: How many
  - Denominator: What: think of as how big

- 4 of 7 parts are shaded
  - Numerator is 4
  - Denominator is 7
Example

\[
\frac{3}{4}
\]

The 3 is how many: Numerator
The 4 is how big – how many equal parts: Denominator
Give the Numerator and Denominator

\[
\frac{3}{7}
\]

\[
\frac{13}{5}
\]

\[
\frac{11}{1}
\]

\[
\frac{9}{2}
\]
Solution

\[
\frac{3}{7} \quad \text{Num: 3, Denom: 7}
\]

\[
\frac{13}{5} \quad \text{Num: 13, Denom: 5}
\]

\[
\frac{11}{1} \quad \text{Num: 11, Denom: 1}
\]

\[
\frac{9}{2} \quad \text{Num: 9, Denom: 2}
\]
For typing ease, \(\frac{3}{4}\) is written \(3/4\)
Fractions Mean Division

\[ \frac{3}{4} = 3 \text{ divided by } 4 \]

\[ \frac{9}{9} = 1 \text{ since } 9 \times 1 = 9 \]

\[ \frac{11}{1} = 11 \text{ since } 11 \times 1 = 11 \]
What about Zero???

0 = 0. There is nothing to count and 0 x 6 = 0

6

6 = ??? This makes no sense. How can you have 6 parts of 0 something that has no size, no parts?

Also, what times 0 = 6????

For any number $n$:

$0/n = 0$;

$n/0$ is undefined – it is not a number
Mnemonic

If k and n are numbers:

- 0/k is OK
- n/0 is NO; no number
Write the Fraction Shaded

[Diagram of a syringe with 10 parts shaded, indicating a fraction]

3 parts
Solution

\[ \frac{3}{10} \]

\[ \frac{1}{2} \]
Write the Fraction Shaded
Solution

• There are 16 circles. 11 are shaded

• Numerator is 11, denominator is 16

• \(\frac{11}{16}\) are shaded
Improper Fractions and Mixed Numbers

In an improper fraction, the numerator is greater than or equal to the denominator, for example $\frac{5}{4}$

$\frac{4}{4} = 1 \quad + \quad \frac{1}{4} = \frac{5}{4} = 1 \ 1/4$

$1 \ 1/4$ is called a mixed number; it has a whole number and a fraction
Examples

Proper Fraction, Improper Fraction or Mixed Number?

a. 6/7
b. 13/12
c. 2/2
d. 99/101
e. 1 7/8
f. 93/74
Solution

Proper Fraction, Improper Fraction or Mixed Number?

a. $\frac{6}{7}$ Proper
b. $\frac{13}{12}$ Improper
c. $\frac{2}{2}$ Improper
d. $\frac{99}{101}$ Proper
e. $1 \frac{7}{8}$ Mixed
f. $\frac{93}{74}$ Proper
Mixed Numbers as Fractions

- For any mixed number, we have a whole number and a fraction.
- A whole number can be written as the number of parts in the whole.

  If we want to write 2 in terms of 1/2, we have two 1/2s in each whole, or $2 \times 2 = 4$ halves total.

  If each small rectangle is 1/2, each large one is 2 halves, and we have 4 halves total = 4/2.
Rule: Mixed Number to Fraction

Step 1: Multiply the denominator by the whole number

Step 2: Add the numerator to the result of Step 1

Step 3: Write the result of Step 2 as the numerator of the improper fraction

1 2/3 = ?

Step 1: 3 x 1 = 3

Step 2: 2 + 3 = 5

Step 3: 5/3

1 2/3 = 5/3
Examples

a. 2 5/7

b. 5 1/3

c. 9 3/10

d. 1 1/5
Solution

a. $2 \frac{5}{7} = \frac{19}{7}$

b. $5 \frac{1}{3} = \frac{8}{3}$

c. $9 \frac{3}{10} = \frac{93}{10}$

d. $1 \frac{1}{5} = \frac{6}{5}$
Rule: Fraction to Mixed Number

Step 1: Divide the numerator by the denominator

Step 2: The whole number is the quotient, the remainder is the numerator which is written over the denominator

\[
\frac{5}{3} = ?
\]

Step 1: \(5 \div 3 = 1\) remainder 2

Step 2: \(\frac{5}{3} = 1 \frac{2}{3}\)

\[
\frac{5}{3} = 1 \frac{2}{3}
\]
Examples

a. \( \frac{9}{5} = \)

b. \( \frac{23}{9} = \)

c. \( \frac{48}{4} = \)

d. \( \frac{62}{13} = \)

e. \( \frac{51}{7} = \)

f. \( \frac{21}{20} = \)
Solution

a. \( \frac{9}{5} = 1 \frac{4}{5} \)

b. \( \frac{23}{9} = 2 \frac{5}{9} \)

c. \( \frac{48}{4} = 12 \)

d. \( \frac{62}{13} = 4 \frac{10}{13} \)

e. \( \frac{51}{7} = 7 \frac{2}{7} \)

f. \( \frac{21}{20} = 1 \frac{1}{20} \)
Summary of Terms

- Fraction
- Numerator
- Denominator
- Proper Fraction
- Improper Fraction
- Mixed Number
Factors and Prime Factorization
What is a Factor?

- A factor of a number is a second number that divides the first number evenly:
  - 2 is a factor of 4
  - 9 is a factor of 18
  - 10 is a factor of 100
  - 100 is not a factor of 10!!!

- What are the factors of 24?
  - 1, 2, 3, 4, 6, 8, 12, 24
Prime Numbers

• A prime number is a number that has only two unique factors (1 and itself, but note that the definition excludes 1 as prime!)

• Prime Numbers: 2, 3, 5, 7, 11, 13, 17…

• What is the next prime number?

• Numbers that are not prime are composite
Prime or Composite?

- 15
- 23
- 37
- 49
- 51
Solution

• $15 = 5 \times 3$, composite

• 23 prime

• 37 prime

• $49 = 7 \times 7$, composite

• 51 prime
Prime Factorization

- Prime factorization means writing a number in terms of its factors, where each factor is prime

- $15 = 3 \times 5$

- $21 = 3 \times 7$

- $25 = 5 \times 5$

- $24 = 2 \times 2 \times 2 \times 3$
Factor Trees:

- Find the prime factorization of 24:
  
  \[ 24 = 3 \times 8; \text{ 3 is prime} \]
  
  \[ 8 = 2 \times 4; \text{ 2 is prime} \]
  
  \[ 4 = 2 \times 2; \text{ 2 is prime} \]
  
  \[ 24 = 2 \times 2 \times 2 \times 3 \]
Find the Prime Factorization

- 30
- 56
- 72
- 117
Solution

- $30 = 2 \times 3 \times 5$

- $56 = 2 \times 2 \times 2 \times 7$

- $72 = 2 \times 2 \times 2 \times 3 \times 3$

- $117 = 3 \times 3 \times 13$
Simplest Form of a Fraction
Equivalent Fractions

• Equivalent fractions are fractions that have the same value

• Examples:
  \[ \frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{5}{10} \]

  \[ \frac{4}{5} = \frac{8}{10} = \frac{20}{100} \]

• Since \( n \times 1 = n \), for any number \( n \),
  and \( \frac{k}{k} = 1 \) for any number \( k \)

  We can create equivalent fractions by multiplying or dividing
  by \( \frac{k}{k} \)
Simplifying Fractions

To simplify a fraction, divide out any factors that are in common in the numerator and the denominator.

Step 1: Find the prime factorization of the numerator and the denominator.

Step 2: Remove common factors from the numerator and denominator. In other words, remove factors of 1 from the fraction.

Example:

\[
\frac{3}{6} = \frac{3}{2 \times 3} = \frac{1}{2} \times \frac{3}{3} = \frac{1}{2} \times 1 = \frac{1}{2}
\]
Examples: Write in Simplest Form

a. $\frac{6}{60} =$

b. $\frac{72}{26} =$

c. $\frac{30}{108} =$

d. $\frac{7}{16} =$
Solutions

a. \( \frac{6}{60} = \frac{1}{10} \)

b. \( \frac{72}{26} = \frac{36}{13} \)

c. \( \frac{30}{108} = \frac{5}{18} \)

d. \( \frac{7}{16} = \frac{7}{16} \)
Multiplying Fractions and Mixed Numbers
Multiplying Fractions

• 2 x 1/2 =

\[
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
= 
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
= 1
\]

• 1/2 x 1/2 = read as 1/2 of 1/2

\[
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
= 1/4
\]
Rule

\[
\begin{array}{ccc}
\frac{a}{b} \times \frac{c}{d} &=& \frac{a \times c}{b \times d}
\end{array}
\]

Examples:

\[
\frac{1}{3} \times \frac{2}{5} = \frac{2}{15}
\]

\[
\frac{2}{3} \times \frac{5}{11} = \frac{10}{33}
\]

Usually write in simplest form
Examples

a. \( \frac{3}{8} \times \frac{5}{7} = \)

b. \( \frac{1}{3} \times \frac{1}{6} = \)
Solutions

a. \( \frac{3}{8} \times \frac{5}{7} = \frac{15}{56} \)

b. \( \frac{1}{3} \times \frac{1}{6} = \frac{1}{18} \)
With Mixed Numbers

Same as fractions, but first change the mixed number to a fraction

Examples:

2 \frac{1}{3} \times \frac{1}{4} = \frac{7}{3} \times \frac{1}{4} = \frac{7}{12}

3 \frac{1}{3} \times \frac{7}{8} = \frac{10}{3} \times \frac{7}{8} = \frac{70}{24} (= \frac{35}{12} = 2 \frac{11}{12})
Examples

a. \( \frac{2}{3} \times 18 = \)

b. \( 2 \frac{1}{2} \times \frac{8}{15} = \)

c. \( 3 \frac{1}{5} \times 2 \frac{3}{4} = \)

d. \( 5 \times 3 \frac{11}{15} = \)

e. \( 0 \times \frac{9}{11} \)
Solutions

a. \( \frac{2}{3} \times 18 = 12 \)

b. \( 2 \frac{1}{2} \times \frac{8}{15} = \frac{4}{3} \)

c. \( 3 \frac{1}{5} \times 2 \frac{3}{4} = \frac{44}{5} \)

d. \( 5 \times 3 \frac{11}{15} = \frac{56}{3} \)

e. \( 0 \times \frac{9}{11} = 0 \)
Dividing Fractions and Mixed Numbers
Dividing Fractions

- $\frac{1}{2} \div 2 = \frac{1}{4}$

  
  ![Diagram showing division of fractions: 1/2 divided by 2 equals 1/4.]

  Think: How many 2s in 1/2? Just 1/4

- $\frac{1}{2} \div \frac{1}{2} = 1$

  ![Diagram showing division of fractions: 1/2 divided by 1/2 equals 1.]

  Think: How many 1/2s in 1/2? Just 1
Rule for Dividing Fractions

Step 1: Invert the second fraction

Step 2: Multiply the first fraction by the inverse

(Invert and Multiply)

Inverses:
Inverse of $\frac{5}{4}$ is $\frac{4}{5}$
Inverse of $\frac{9}{8}$ is $\frac{8}{9}$
Inverse of $\frac{12}{5}$ is $\frac{5}{12}$
Division Examples

\[ \frac{4}{7} \div \frac{3}{5} = \]
Inverse of \( \frac{3}{5} \) is \( \frac{5}{3} \)
Multiply: \( \frac{4}{7} \times \frac{5}{3} = \frac{20}{21} \)

\[ \frac{4}{15} \div \frac{2}{3} = \]
Inverse of \( \frac{2}{3} \) is \( \frac{3}{2} \)
Multiply: \( \frac{4}{15} \times \frac{3}{2} = \frac{12}{30} = \frac{2}{5} \)
Mixed Numbers

• Just as with multiplication, make the mixed number a fraction

• Example:

\[3 \frac{1}{4} ÷ \frac{1}{5}\]

\[= \frac{13}{4} ÷ \frac{1}{5}\]

\[= \frac{13}{4} \times 5 = \frac{65}{4}\]
• To multiply fractions: Multiply numerators then multiply denominators: \( \frac{3}{4} \times \frac{2}{3} = \frac{(3 \times 2)}{(4 \times 3)} = \frac{6}{12} = \frac{1}{2} \)
• To multiply mixed numbers: Change to fractions then multiply: \( 1 \frac{1}{2} \times \frac{1}{3} = \frac{3}{2} \times \frac{1}{3} = \frac{3}{6} = \frac{1}{2} \)
• To divide fractions: Invert the second fraction then multiply: \( \frac{1}{2} \div \frac{3}{5} = \frac{1}{2} \times \frac{5}{3} = \frac{5}{6} \)
• To divide mixed numbers, change to a fraction first
• A prime number has only two unique factors, 1 and itself
  − Every composite number can be factored by primes
  − Primes are useful for simplifying fractions (and other things)
Some More Examples

a. \( \frac{4}{9} \div 7 = \)

b. \( \frac{8}{15} \div 3 \frac{4}{5} = \)

c. \( 2 \frac{2}{7} \div 2 \frac{3}{14} = \)

d. \( \frac{14}{17} \div 0 = \)

e. \( 0 \div 2 \frac{1}{8} = \)
Solutions

a. \( \frac{4}{9} \div 7 = \frac{4}{63} \)

b. \( \frac{8}{15} \div 3 \frac{4}{5} = \frac{8}{15} \times \frac{6}{19} = \frac{8}{57} \)

c. \( 2 \frac{2}{7} \div 2 \frac{3}{14} = \frac{46}{31} \)

d. \( \frac{14}{17} \div 0 = \text{undefined} \)

e. \( 0 \div 2 \frac{1}{8} = 0 \)
Summary

• Arithmetic with fractions
  – Multiplying
  – Dividing
  – Mixed Numbers
  – Prime Factoring
  – Simplifying fractions
Adding and Subtracting Fractions
Overview

• Adding and Subtracting Fractions
• Greatest Common Multiple
• Unlike Fractions
• Mixed numbers
• Order, exponents, order of operations
The Key to Adding and Subtracting Fractions

• You have to add and subtract equal objects:
  1/4s and 1/4s, 1/6s and 1/6s
  It’s like adding cats and dogs and monkeys – you can’t do it unless you make them all animals, all mammals, etc.
  In other words, they have to all be the same

• Once you have equal objects, just add the numerators and simplify the fraction
  1/4 + 3/4 = 4/4 = 1
  3/5 + 1/5 = 4/5
Rule

Step 1: Find a common denominator

Step 2: Use equivalent fractions to make the fractions have the same denominator

Step 3: Add or subtract the numerators

Step 4: Simplify

$\frac{1}{4} + \frac{1}{2} =$

Step 1: denominator = 4

Step 2: $\frac{1}{4} + \frac{2}{4}$

Step 3: = 3/4

$\frac{2}{5} - \frac{1}{10} = \frac{4}{10} - \frac{1}{10} = \frac{3}{10}$
Finding the Common Denominator

Need to find a number that is a multiple of both denominators

1/5 and 1/6? Need 1/30
3/4 and 1/2? Need 1/4 (or 1/8)
2/9 and 3/2? Need 1/18

How to find?

1. Can always multiply the denominators, but that can be a big number and create extra work
   Ideally, you want to work with the smallest number possible
2. Find the least common multiple through prime factoring
Finding the Least Common Multiple (LCM)

Say have denominators of 12 and 18

\[ 12 = 2 \times 2 \times 3 \]

\[ 18 = 2 \times 3 \times 3 \]

To be a multiple of both denominators, need \( 2 \times 2 \times 3 \times 3 = 36 \)

\[ 12 \times 3 = 36 \]

\[ 18 \times 2 = 36 \]

Note: Sometimes you can figure it out in your head – feel free
Examples: Find the LCM

• 8 and 10

• 8 and 16

• 25 and 30

• 20, 24 and 45
Solutions

- 8 and 10; 40
- 8 and 16; 16
- 25 and 30; 150
- 20, 24 and 45; 360
Add the Fractions

- \( \frac{1}{6} + \frac{5}{18} = \)

- \( \frac{5}{6} + \frac{2}{9} = \)

- \( \frac{2}{5} + \frac{4}{9} \)

- \( \frac{7}{12} - \frac{5}{24} \)

- \( \frac{9}{10} - \frac{3}{7} \)
Solution

• $\frac{1}{6} + \frac{5}{18} = \frac{4}{9}$

• $\frac{5}{6} + \frac{2}{9} = \frac{19}{18}$

• $\frac{2}{5} + \frac{4}{9} = \frac{38}{45}$

• $\frac{7}{12} - \frac{5}{24} = \frac{3}{8}$

• $\frac{9}{10} - \frac{3}{7} = \frac{33}{70}$
Mixed Number

Step 1: Add or subtract the fractions

Step 2: Add or subtract the whole numbers

Step 3: Combine and simplify

$$4 \ 2/5 + 5 \ 1/6 =$$

$$4 + 5 = 9$$

$$2/5 + 1/6 = 12/30 + 5/30 = 17/30$$

$$= 9 \ 17/30$$
Subtraction

- Follow same steps, but

  \[ 3 \frac{1}{4} - 1 \frac{1}{2} = ? \]

Fractions: \( \frac{1}{4} - \frac{1}{2} = \frac{1}{4} - \frac{2}{4} = -\frac{1}{4} \)

Whole Numbers: \( 3 - 1 = 2 \)

Combining: How do we combine 2 and \(- 1 \frac{1}{4}\)?

Make the 2 = 1 + \( \frac{4}{4} \) then combine: \( 2 - 1 + \frac{4}{4} - \frac{1}{4} \)

\[ 3 \frac{1}{4} - 1 \frac{1}{2} = 1 \frac{3}{4} \]
Examples:

a. \( 4 \frac{2}{5} + 5 \frac{1}{6} = \)

b. \( 2 \frac{5}{14} = 5 \frac{6}{7} \)

c. \( 10 + 2 \frac{6}{7} + 3 \frac{1}{5} = \)

d. \( 29 \frac{7}{9} - 13 \frac{5}{18} \)

e. \( 9 \frac{7}{15} - 5 \frac{3}{5} \)
Solutions

a. $4 \frac{2}{5} + 5 \frac{1}{6} = 9 \frac{17}{30}$

b. $2 \frac{5}{14} = 5 \frac{6}{7} = 8 \frac{3}{14}$

c. $10 + 2 \frac{6}{7} + 3 \frac{1}{5} = 16 \frac{2}{35}$

d. $29 \frac{7}{9} - 13 \frac{5}{18} = 16 \frac{1}{2}$

e. $9 \frac{7}{15} - 5 \frac{3}{5} = 3 \frac{13}{15}$
Only One Whole Number or Fraction?

Treat the other as zero

- $3 + 1 \frac{1}{2} = 4 \frac{1}{2}$
- $\frac{1}{2} + 2 \frac{1}{4} = 2 \frac{3}{4}$
- $34 \frac{1}{5} - 1\frac{1}{10} = 34 \frac{1}{10}$
- $25 - 10 \frac{2}{9} = 24 \frac{9}{9} - 10 \frac{2}{9} = 14 \frac{7}{9}$
Comparing Fractions

• Which is larger 1/4 or 1/3?

  – The larger the number of pieces, the smaller each piece is

  – Can always make sure by getting a common denominator

• The Hamburger Fiasco (why you need to study math)
Examples: Which is Larger

a. 1/7 or 1/9

b. 8/9 or 10/11

c. 3/5 or 2/9
Solution: Which is Larger

a. 1/7 or 1/9: 1/7

b. 8/9 or 10/11: 10/11

c. 3/5 or 2/9: 3/5
Raising Fractions to Powers

• \( a^2 = a \times a \)

• \( b^4 = b \times b \times b \times b \)
  
  *We can always count the number of terms*

• \( (a/b)^2 = (a/b) \times (a/b) = (axa) / (bxb) = a^2/b^2 \)
Examples

a. \( \left( \frac{1}{5} \right)^2 = \)

b. \( \left( \frac{2}{3} \right)^3 = \)

c. \( \left( \frac{1}{4} \right)^2 \left( \frac{2}{3} \right)^3 = \)
Solutions

a. \((1/5)^2 = 1/25\)

b. \((2/3)^3 = 8/27\)

c. \((1/4)^2 (2/3)^3 = 1/54\)
Order of Operations

Think of this as mathematical grammar

1. Perform everything inside parentheses
2. Evaluate exponents
3. Multiply or divide, left to right
4. Add or subtract, left to right

PEMDAS: Please Excuse My Dear Aunt Sally
   P – parentheses, E – exponents, M & D – multiply and divide
   A & S – add and subtract
Examples

a. \( \frac{2}{9} \div \frac{4}{7} \times \frac{3}{10} = \)

b. \( \left(\frac{2}{5}\right)^2 \div \left(\frac{3}{5} - \frac{11}{25}\right) = \)
Solution

a. \( \frac{2}{9} \div \frac{4}{7} \times \frac{3}{10} = \frac{7}{60} \)

b. \( \left(\frac{2}{5}\right)^2 \div \left(\frac{3}{5} - \frac{11}{25}\right) = 1 \)
Summary

• Arithmetic with fractions
  – Multiplying
  – Dividing
  – Mixed Numbers
  – Prime Factoring
  – Simplifying fractions
Fraction Review

1) \( \frac{4}{5} \times \frac{3}{4} = \)

2) \( \frac{2}{3} \div \frac{5}{2} = \)

3) \( 3 \frac{1}{4} \times 2 \frac{2}{3} = \)

4) \( 5 \frac{1}{6} \div 3 \frac{1}{3} = \)

5) \( \frac{3}{4} + \frac{2}{9} = \)

6) \( \frac{5}{7} - \frac{2}{3} = \)

7) \( 2 \frac{1}{4} + 3 \frac{1}{5} = \)

8) \( 3 \frac{1}{3} - 2 \frac{4}{5} = \)

9) \( 1 \frac{1}{2} - 2 \frac{1}{3} = \)

10) Simplify \( \frac{65}{52} \)